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Pure Mathematics 3

Information Bulletin

2001–2002 School Year

Diploma Examinations Program

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Bulletin Information

The purpose of this bulletin is to provide students and teachers of Pure Mathematics 30 with information about the diploma examinations scheduled in the school year. This bulletin should be used in conjunction with the 2001–02 *Pure Mathematics Program of Studies* to ensure that curriculum and standards are addressed.

This bulletin includes descriptions of the Pure Mathematics 30 diploma examinations that will be administered in November 2001, January, June, and August of 2002, descriptions of the *acceptable standard* and the *standard of excellence*, and subject-specific information. The mark awarded to a student on the Pure Mathematics 30 diploma examinations in November 2001, January, June, and August of 2002 will account for 50% of the student's final blended mark and the school-awarded mark will account for the remaining 50%.

Teachers are encouraged to share the contents of this bulletin with students.

A Mathematics 30 diploma examination will continue to be available in all regular writing administrations until August 2002, except for November 2001. All of these examinations will be secured (i.e., not released to educators, students, or the general public after the administration). These examinations are intended for repeating students, adult students, distance learning students, and students with special circumstances.

For further information about program implementation refer to the Alberta Learning web site at http://www.learning.gov.ab.ca.

Objectives of the Course

The Pure Mathematics 30 course emphasizes mathematical theory. In pure mathematics, algebra and graphing are used to solve problems. Deductive and symbolic methods are used to determine if and under what conditions a concept is true.

Students are expected to communicate solutions to problems clearly and effectively when solving both routine and non-routine problems. Technology is to be used for exploration, modeling, and problem solving. Students are also expected to apply mathematical concepts and procedures to real-life problems.

The Pure Mathematics 30 course consists of two sets of outcomes, as specified in the *Program of Studies*. One set of outcomes is common to both the pure and applied courses and the second set of outcomes is for the pure course only.

Standards

Curriculum Standards

Provincial curriculum standards help to communicate how well students need to perform in order to be judged as having achieved the learnings specified for Pure Mathematics 30. These specific statements of standards are written primarily to apprise Pure Mathematics 30 teachers of the extent to which students must know the Pure Mathematics 30 content and must demonstrate the required skills to be able to pass the examination.

Examples of Questions

In this bulletin, there are examples of questions that students should be able to answer in order to demonstrate the *acceptable standard* and the *standard of excellence*. The examples provided are by no means exhaustive; they are intended to provide a profile of acceptable and excellent achievement. Some of the examples and solutions were developed and validated by classroom teachers of mathematics but have not been validated with students. They model the types of questions and problems that students should be able to solve and the types of activities that students should be able to perform in order to meet the specific outcome to which they are linked. Some examples included in the 2001–02 *Bulletin* have been validated with students.

Achievement Standards

Acceptable Standard

Students who attain the *acceptable standard* but not the standard of excellence in Pure Mathematics 30 will receive a final course mark between and including 50% and 79%. Typically, these students have gained new skills and knowledge in mathematics and they can apply mathematical concepts and procedures to find a solution to routine problems. They have demonstrated mathematical skills and knowledge in the seven content strands of the Pure Mathematics 30 curriculum and exhibit an ability to apply a broad range of problem-solving skills to these content strands.

Standard of Excellence

Students who attain the *standard of excellence* will receive a final course mark of 80% or higher. Such students have demonstrated their ability and interest in mathematics and have confidence in their mathematical abilities. These students can choose the most efficient method for solving problems. They can also find more than one solution and can solve non-routine problems.

Achievement Standards

At a provincial level, it is expected that at least 85% of students will achieve a final course mark of 50% or higher and that at least 15% of students will achieve a final course mark of 85% or higher.

Term Project

Alberta Learning will produce two projects (large-scale mathematical problems) per year for Pure Mathematics 30. These projects are designed to be completed in three to five hours of student time. Use of these projects is optional. Teachers may choose to use the projects as part of their assessment. A sample solution and scoring rubric will be provided for each project. One of the written-response questions worth 10% on the diploma examination will be related to its corresponding project. The first project will be distributed to schools in mid-August 2001, and one written-response question on each of the November 2001 and January 2002 diploma examination will be related to this project. The second project will be distributed to schools in mid-January 2002 and one written-response question on each of the June 2002 and August 2002 diploma examinations will be related to this project. Students who did not complete the project but who have completed the course will have the knowledge to answer the written-response question; however, students who have completed the project will have had experience with the related mathematical skills in another context.

Examination Specifications

Each Pure Mathematics 30 Diploma Examination is designed to reflect the core content outlined in the *Pure Mathematics 30 Program of Studies*. The examination is limited to those expectations that can be measured by a paper-and-pencil test. Therefore, the percentage weightings shown below will not necessarily match the percentage of class time devoted to each unit. The examination is developed to be completed in 2.5 h; however, students may take an additional 0.5 h to complete the examination.

Students are expected to provide a graphing calculator approved by Alberta Learning and students are also expected to have cleared their calculator of all information that is stored in the programmable or parametric memory.

The content for the Pure Mathematics 30 diploma examinations in the 2001–02 school year is emphasized as follows:

Machine-Scored Content	*Percent Emphasis*	
Transformations of Functions	15	
Exponents, Logarithms, and Geometric Series	18	
Trigonometry	22	**NEW**
Conic Sections	12	
Permutations and Combinations	13	
Statistics, Probability	20	
Multiple-Choice and Numerical-Response Questions	65	
Written-Response Questions	35	

Written-response questions assess the degree to which students can draw on their mathematical experiences to solve problems and explain mathematical concepts. Therefore, the written-response questions will not necessarily fall into a particular unit of study but may cover more than one unit or may require students to make the connections between mathematical concepts. Of the seven mathematical processes, the written-response questions will address communication, connections, reasoning, problem-solving, and technology.

Procedural, conceptual, and problem solving cognitive levels are addressed throughout the examination. The approximate emphasis of each cognitive level is given below.

Multiple Choice, Numerical Response, and Written Response	***Percent Emphasis***
Procedures	34
Concepts	33
Problem Solving	33

Since the machine-scored and written-response questions are weighted components, a student's examination mark can be calculated using the following equation:

$$\left(\frac{x}{39} \times 65\right) + \left(\frac{y}{10} \times 20\right) + \left(\frac{z}{5} \times 15\right) = \text{exam mark}$$

x = Student's total score on multiple-choice and numerical-response questions

y = Student's total score on the two 10% written-response questions

z = Student's score on the 15% written-response question.

Examination Design

The design of the 2001–02 Pure Mathematics 30 diploma examinations is as follows:

Question Format	*Number of Questions*	*Percent Emphasis*
Multiple Choice	33	55
Numerical Response	6	10
Written Response	3	35

The six numerical-response questions are interspersed throughout the multiple-choice questions, according to content topic.

The three written-response questions appear together following the machine-scored questions. You may wish to encourage your students to look at these questions first.

Information required to answer multiple-choice and/or numerical-response questions is often located in a box preceding the question. The number of questions that require the use of the information given in the box will be clearly stated above the box; e.g., "*Use the following information to answer the next two questions.*"

Multiple Choice and Numerical Response

For multiple-choice questions, students are to choose the correct or best possible answer from four alternatives.

For some numerical-response questions, students are to calculate a numerical answer and to record their answer in a separate area of the answer sheet. When the answer to be recorded cannot be a decimal value, students are asked to determine a whole number value (e.g., the number of people is _____, the degree of this polynomial is _____.) If the answer can be a decimal value, then students are asked to record their answer correct to the nearest tenth or nearest hundredth.

Other numerical-response questions require students to record their understanding of a conceptual idea. The following is an example of such a question.

Use the following information to answer the next question.

The three expressions below each have a particular value.

1 ${}_7C_2$

2 ${}_5P_2$

3 $4!$

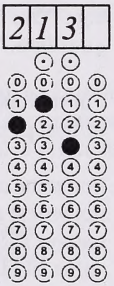
Numerical Response

6. If the expressions above are ordered from lowest value to highest value, then their corresponding expression numbers will be ____, ____, and ____.

(Record **all three digits** of your answer in the numerical-response section on the answer sheet.)

Answer: 213

Record 213 on the answer sheet →



Written Response

The written-response questions focus on students' understanding of the process of solving a problem and encourage students to take risks to arrive at a solution. Students will be rewarded for selecting a problem-solving strategy and for carrying through with the strategy to find a solution. The written-response questions focus on the students' use of mathematical processes. To achieve the *standard of excellence*, students must be able to select a strategy, carry it through, and solve the problem correctly. The written-response questions also focus on students' understanding of mathematical concepts and allow for flexibility in evaluating students' communication and problem-solving abilities in mathematics.

Written-response questions ask students to solve, explain, justify, or prove their solution.

In scoring the written-response questions, markers will evaluate how well students

- understand the problem or the mathematical concept
- correctly use the mathematics
- use problem-solving strategies and explain their answer and procedures
- communicate their solutions and mathematical ideas

One of the two 10% questions will be related to the term project provided by Alberta Learning to all schools. The 15% question will focus on the mathematical processes of reasoning and connections.

The written-response questions on the diploma examination will appear in the following order. First, a 10% written-response question related to the project; second, a 15% written-response question; third, a 10% written-response question which is the last question on the diploma. **NEW**

Above all, students should be encouraged to try to solve all problems. Even an attempt at a solution could be worth some marks. The three written-response questions are each scored with a five-mark scoring rubric. Students should note that markers will be instructed to consider appropriate use of units and appropriate rounding of answers in all solutions to written-response questions. A student will not be penalized more than once for errors of this type.

Assessment of the Mathematical Processes

Open-ended questions provide a way in which to assess mathematics as a common human activity. Open-ended questions allow students to communicate a response by asking them to explain their reasoning, explain their solution, describe mathematical situations, write directions, create new problems, create new strategies, generalize a mathematical situation, and formulate hypotheses.

The written-response questions provide an opportunity to evaluate several of the seven mathematical processes: communication, connections, estimation and mental mathematics, problem-solving, reasoning, technology, and visualization. One of the 10% written-response questions will focus on problem-solving, and the other 10% question will focus on communication. The 15% written-response question will focus on connections and reasoning. Technology may be incorporated into all three written-response questions.

Communication

"Students need to communicate mathematical ideas clearly and effectively, orally and in writing." (*Applied and Pure Mathematics Program of Studies,* page 6) It is expected that students, in answering the written-response questions, will provide clear, concise answers. Students should clearly communicate all pertinent steps used to arrive at a solution. Students should use correct mathematical notation and conventions and graphs or diagrams may be presented in solutions.

In communicating an answer, students must be aware of the degree of accuracy required as well as the appropriate units involved.

Problem Solving

"Problem-solving is the focus of mathematics at all grade levels. The development of each student's ability to solve problems is essential. Students develop a true understanding of mathematical concepts and procedures when they solve problems in meaningful contexts." (*Applied and Pure Mathematics Program of Studies,* page 8) Approximately one-third of the machine-scored questions as well as one of the written-response questions will be related to a problem solving context. It is expected that students will be able to select an appropriate problem-solving strategy to find the solution to a given problem or to model real-world problem situations.

Students should not expect questions on a particular concept to be asked in the same manner every time. They must be able to adapt to changes in the format, which will help to improve their problem-solving skills.

Connections

Students need numerous and varied experiences in order to appreciate the usefulness of mathematics. They must explore connections between different areas of mathematics and between mathematics and other disciplines. The "connections process" also includes relating mathematics to their own daily experiences. "When mathematical ideas are connected to each other through concrete, pictorial, and symbolic representations, students begin to view mathematics as an integrated whole." (*Applied and Pure Mathematics Program of Studies,* page 7)

The connections process is often linked with problem-solving and reasoning as it is an application of these other processes.

Reasoning

"Students need to develop confidence in their ability to reason and to justify their thinking within and outside of mathematics. The power of reasoning helps students to make sense of mathematics, to make logical inferences from given information, and to convince others of a mathematical meaning.

"Inductive reasoning helps students explore and make conjectures from activities that allow generalizations from a pattern of observations.

"Deductive reasoning helps students test conjectures and build arguments that serve to validate logical thinking. Deductive reasoning also assembles a structured body of knowledge." (*Applied and Pure Mathematics Program of Studies,* page 9)

Reasoning allows students to interpret mathematics and aids in the choice of appropriate problem-solving strategies. Students may be asked to demonstrate logical reasoning when judging the validity of arguments, testing conjectures, and constructing valid arguments.

Technology

"Electronic technologies—calculators and computers—are tools that furnish visual images of mathematical ideas, facilitate organization and analysis of data, and support investigation by students. They allow students to focus on decision making, reflection, reasoning, and problem solving. The computational capacity of technological tools extends the range of problems accessible to students as well as enabling them to execute routine procedures quickly and accurately, thus allowing more time for conceptualizing and modeling. Technology also blurs some of the separations among topics in algebra, geometry, and data analysis by allowing students to connect knowledge and ideas from one area of mathematics to another." *(Principles and Standards for School Mathematics, NCTM 2000,* pages 24 to 25*).*

General Scoring Guide

Scoring Guide for Written-Response Questions

Credit may be given to students who show unusual insight. If their solutions fall outside *specific question scoring rubrics*, they will be scored against the *General Scoring Guide* shown on the next page.

A 'five' need not be a perfect paper!

This scoring guide reflects a mark based on four dimensions:

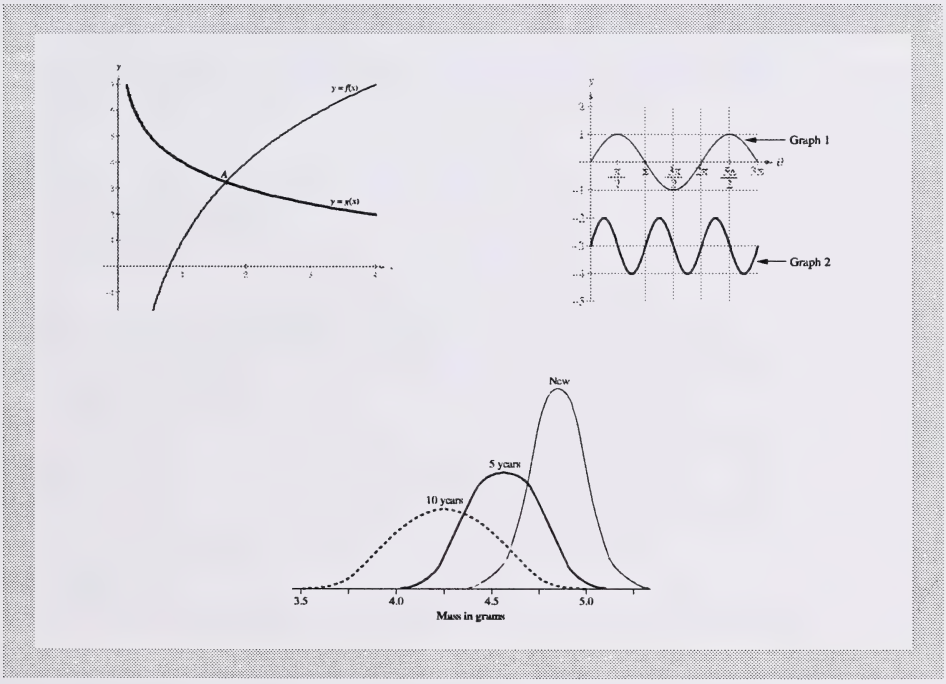
- mathematical understanding
- application of processes
- clarity of communication
- use of technology

GENERAL SCORING GUIDE	
1 mark	In the response, the student • applies some relevant mathematical knowledge to explore the initial stages of the problem; however, the response reflects a misunderstanding of the problem • uses a relevant strategy, mathematical process, or problem-solving technique to explore the initial stages of the problem • communicates very little relevant information and lacks clarity • uses technology inappropriately or the use of technology is not evident
2 marks	In the response, the student • applies some relevant mathematical knowledge to find partial solutions to the problem; however, the response reflects a minimal understanding of the problem • uses relevant strategies, mathematical processes, or problem-solving techniques to find a partial solution to the problem • communicates strategies in a manner that lacks clarity or is incomplete • uses technology where appropriate; however, errors are evident
3 marks	In the response, the student • applies mathematical knowledge to find partial solutions to the problem and reflects a basic understanding of the problem • uses appropriate strategies, mathematical processes, and problem-solving techniques to find partial solutions to the problem • communicates the strategies and solutions in an organized manner; however, errors, inconsistencies, and omissions affect clarity • uses technology appropriately; however, there are inconsistencies in their application
4 marks	In the response, the student • applies appropriate mathematical knowledge to find a complete solution to the problem and reflects a good understanding of the problem • uses appropriate strategies, mathematical processes, and problem-solving techniques to find a complete solution to the problem; however, the solution contains an error that hinders understanding of the response • communicates strategies and solutions in an organized manner; however, errors or omissions may affect clarity • uses technology appropriately
5 marks	In the response, the student • applies appropriate mathematical knowledge to find a complete and correct solution to the problem and reflects an excellent understanding of the problem • uses appropriate strategies, mathematical processes, and problem-solving techniques to find a complete, correct solution; the solution may have a minor error but it does not hinder the understanding of the response • communicates strategies and solutions in a clear, complete, and organized manner that reflects a thorough understanding of the problem • uses technology effectively

Pure Mathematics 30

Curriculum Standards and Example Questions

June 2001



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Many of the examples shown in this document were chosen to illustrate the intent of particular outcomes of Pure Mathematics 30, but will not necessarily be assessed on a diploma examination in the manner shown. Some examples were developed and validated by classroom teachers of mathematics but have not been validated with students. Other examples were taken from the January 2001 Diploma Examination.

To meet the outcomes of Pure Mathematics 30, students will need access to an approved graphing calculator and a computer with a spreadsheet program. In most classrooms, students will use a graphing calculator on a daily basis. Refer to the Alberta Learning web site www.learning.gov.ab.ca for a list of approved graphing calculators.

This document presents a draft revision of the curriculum standards. If you have comments or questions regarding this document, please forward them by e-mail to Hank Reinbold (Hank.Reinbold@gov.ab.ca) phone (780) 427-0010 or fax (780) 422-4200.

The Learner Assessment Branch would like to recognize and thank the many teachers throughout the province who helped prepare this document. We would also like to thank the Curriculum Standards Branch and the Learning Technologies Branch for their input and assistance in reviewing these standards.

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Transformations

General Outcome

Perform, analyze, and create transformations of functions and relations that are described by equations or graphs.

General Notes:

- Analysis of domain, range, intercepts, and asymptotes will be addressed in these outcomes.
- Approximately 80% of the emphasis in this unit should be on single transformations of functions, 10% of the emphasis should be on combined transformations of functions, and 10% of the emphasis should be on transformations of relations.
- A translation of the point (x, y) to the point $(x - h, y - k)$ is to be considered a single transformation, not a combined transformation.
- The use of the notation $y - k = f(x)$ is intended to emphasize that this is a transformation on y; this notation is particularly useful when transforming conic sections. This notation can be expressed as $y = f(x) + k$ for use on the graphing calculator.
- Students should be able to perform, analyze, and describe the transformation of any relation given its graph. However, algebraic transformations should be restricted to functions that have been previously studied; i.e., polynomial functions, rational functions, absolute value functions, and radical functions. By the end of the Pure Mathematics 30 course, students should be able to perform, analyze, and describe transformations on the additional functions and relations studied in the course; i.e., trigonometric functions, logarithmic functions, quadratic functions (conic sections), and exponential functions.

Specific Outcomes

Specific Outcome P9-4

Describe how various translations of functions affect graphs and their related equations.

- $y = f(x - h)$
- $y - k = f(x)$

[C, T, V]

P9-4 Note:

- Notation for translations appears in the resources as $y = f(x - h)$ and $y - k = f(x)$, or as the combined form $y - k = f(x - h)$.

Specific Outcome P9-5

Describe how various stretches of functions (compressions and expansions) affect graphs and their related equations.

- $y = af(x)$ or $\left(\frac{1}{a}\right)y = f(x)$
- $y = f(bx)$

[C, T, V]

P9-5 Notes:

- Students must be able to indicate the line about which a given function is stretched vertically or horizontally.
- If there is a vertical stretch, then the factor is a, and if there is a horizontal stretch, then the factor is $\frac{1}{b}$.
- If the $|\text{factor}| < 1$, then there is a compression and if the $|\text{factor}| > 1$, then there is an expansion.
- Students may find it useful to describe these transformations in this manner: "The graph of the function is stretched vertically/horizontally about the line _____ by a factor of _____, therefore, it is an expansion/compression." **NEW**

(See example 4)

Specific Outcome P9-6

Describe how reflections of functions in both axes and in the line $y = x$ affect graphs and their related equations.

- $y = f(-x)$
- $y = -f(x)$
- $y = f^{-1}(x)$

[C, T, V]

P9-6 Note:

- Note that $y = f^{-1}(x)$ is equivalent to $x = f(y)$, the inverse of $y = f(x)$, but not to the reciprocal function.

(See example 7)

Specific Outcome P9-7

Using the graph and/or the equation of $f(x)$, describe and sketch $\frac{1}{f(x)}$. [C, T, V]

P9-7 Note:

- Domain, range, and asymptotes should be discussed. This outcome is a continuation of rational functions from Pure Mathematics 20.

(see examples 5, and 8)

Specific Outcome P9-8

Using the graph and/or the equation of $f(x)$, describe and sketch $|f(x)|$. [C, T, V]

P9-8 Note:

- Domain and range should be discussed.

Specific Outcome P9-9

Describe and perform single transformations and combinations of transformations on functions and relations. [C, T, V]

P9-9 Notes:

- On the diploma examination, transformations of relations will be assessed as part of the conics unit.
- Students should be encouraged to follow the order of operations when being asked to sketch transformations. However, students should realize that the order in which transformations are done can vary (i.e., BEDMAS versus stretches/reflections and then translations) and they should be able to describe these transformations using either order.
- Students must realize that a function must be rewritten as $y = a \cdot f[b(x + h)] + k$ when they draw or describe transformations. This same idea will be used in the analysis of trigonometric functions.

(See examples 1, 2, 3, and 6)

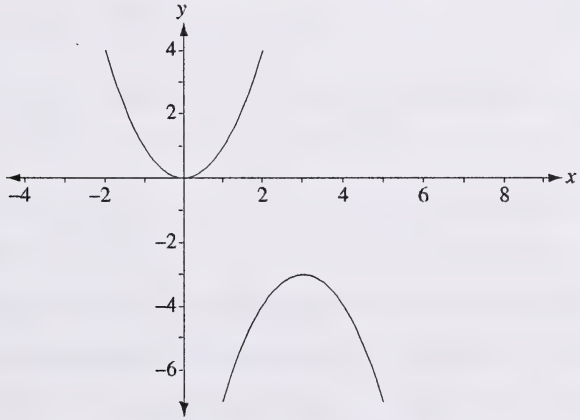
Acceptable Standard	*Standard of Excellence*
The student can	The student can also
• determine and describe the effects of a transformation on the domain, the range, and the intercepts of the relation or function	
• recognize the graph associated with an algebraic function (as listed in the general notes)	• determine the equation of a transformed function given its graph
• use appropriate notation to describe a transformation	
• given the function f in equation or graphical form, perform, analyze, and describe a horizontal shift $(y = f(x - h))$ and/or a vertical shift $(y - k = f(x))$ graphically or algebraically	
• given the function $y = f(x)$ in equation or graphical form, perform, analyze, and describe a horizontal stretch $(y = f(bx)$, where $b > 0$ or a vertical stretch $(y = af(x)$, where $a > 0)$ graphically or algebraically	• given the function $y = f(x)$ in equation or graphical form, perform, analyze, and describe a horizontal stretch $(y = f(bx)$, where $b < 0,\ b \neq -1)$ or a vertical stretch $(y = af(x)$, where $a < 0,\ a \neq -1)$, graphically or algebraically
• given the function f in equation or graphical form, perform, analyze, and describe a reflection in the x-axis $(y = -f(x))$ or in the y-axis $(y = f(-x))$ or in the line $y = x$ $(y = f^{-1}(x))$ graphically or algebraically	
• sketch and describe $y = \frac{1}{f(x)}$ given the equation or graph of $f(x)$	
• analyze domain, range, and asymptotes of $y = \frac{1}{f(x)}$	
• analyze and determine the equation of $y = \frac{1}{f(x)}$, given the graph of $f(x)$	
• sketch and describe $y = \lvert f(x) \rvert$, given the graph or equation of $f(x)$	• analyze and determine the equation of $y = \lvert f(x) \rvert$, given the graph of $f(x)$
• perform, analyze, and describe combinations of transformations on functions or relations not involving reflections	• perform, analyze, and describe combinations of transformations on functions or relations involving reflections

Acceptable Standard	*Standard of Excellence*
• participate in and contribute toward the problem-solving process for problems involving the transformations of functions and relations studied in Pure Mathematics 30	• complete the solution to problems involving the transformations of functions and relations studied in Pure Mathematics 30

Examples

Students who achieve the *Acceptable Standard* should be able to answer all the following questions, except for any part of a question labelled SE. Parts labelled SE are appropriate examples for students who achieve the *Standard of Excellence*.

1. A translated image of the graph of $y = x^2$ is shown below. If its vertex is at $(3, -3)$, what is the equation of the transformed image? Explain your answer.



Solution

Since the graph opens downward, it is a reflection of $y = x^2$ in the x-axis. The graph has been translated horizontally 3 units in the positive x direction and vertically 3 units in the negative y direction; therefore, the equation is $y + 3 = -1(x - 3)^2$ or $y = -1(x - 3)^2 - 3$.

2. Describe the transformations on the graph of the function $g(x) = 2(x-4)^2 + 3$.

Solution

The graph of the function is stretched vertically about the x-axis ($y = 0$) by a factor of 2, therefore it is an expansion. The function is then moved up 3 units and to the right 4 units.

or

The graph of the function is shifted right 4 units, stretched vertically about the x-axis by a factor of 2, therefore it is an expansion. The function is then moved up 3 units. (BEDMAS)

SE **3.** Describe the transformations on the graph of the function $h(x) = -\frac{1}{2}[(3x + 6)]^2 - 1$.

Solution

First rewrite the function as $h(x) = -\frac{1}{2}[3(x + 2)]^2 - 1$. The graph of this function is stretched vertically about the x-axis ($y = 0$) by a factor of $\frac{1}{2}$, therefore it is a compression, then reflected in the x-axis. The graph is then stretched horizontally about the y-axis ($x = 0$) by a factor of $\frac{1}{3}$, therefore it is a compression. The function is and then moved 2 units to the left and 1 unit down.

4. Compare the transformations on the graph of $f(x) = (2x)^2$ and the graph of $g(x) = 4x^2$.

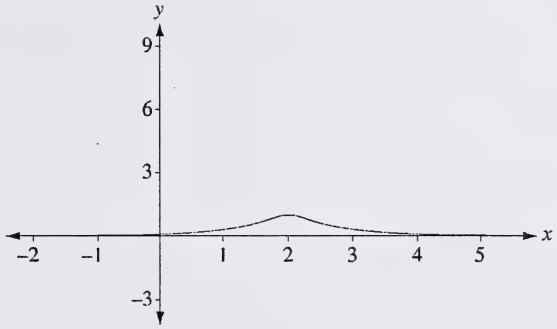
Solution

The graph of $f(x) = (2x)^2$ is stretched horizontally about the y-axis by a factor of $\frac{1}{2}$, therefore is a compression. The graph of $g(x) = 4x^2$ is stretched vertically about the x-axis by a factor of 4, therefore it is an expansion. Although the transformation descriptions are different, the resulting graphs are identical.

SE **5.** The graph of $G(x) = \frac{1}{f(x)}$, where $f(x)$ is a quadratic function, is shown below.

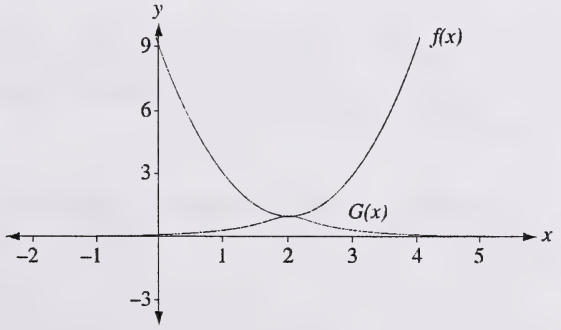
The y-intercept of $G(x)$ is $\frac{1}{9}$, and the maximum point of $G(x)$ is at $(2, 1)$.

What is the equation of $f(x)$?

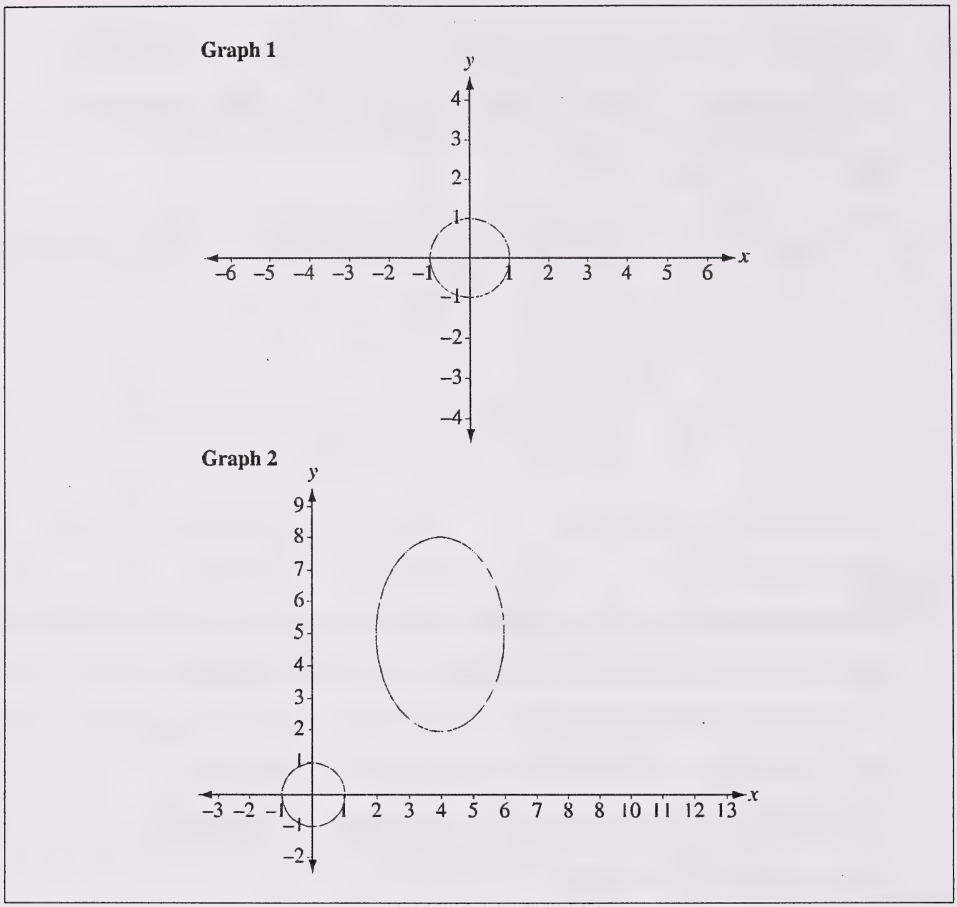


Solution

Sketch the parabola from the given information. The parabola will be symmetrical about the line $x = 2$, and it will have a minimum point at $(2, 1)$. The reciprocal of $\frac{1}{9}$ is 9, so the y-intercept will be 9. The sketch below illustrates that there is a horizontal shift of 2 and a vertical shift of 1; therefore, the equation of this function is $y - 1 = a(x - 2)^2$ or $y = a(x - 2)^2 + 1$. To find the value of a, substitute $(0, 9)$ into the equation. The value of $a = 2$, so the final equation is $y = 2(x - 2)^2 + 1$.



Use the following information to answer the next question.



6. Given the graph of a circle with centre at (0, 0) and radius 1, as shown above in graph 1, describe the series of transformations required to transform graph 1 to graph 2 which is an ellipse with centre at (4, 5), a length of 6 units measured parallel to the *y*-axis, and a width of 4 units measured parallel to the *x*-axis.

Solution

Graph 1 has to be stretched vertically by a factor of 3 about the *x*-axis and stretched horizontally by a factor of 2 about the *y*-axis. The graph is then translated 4 units to the right and 5 units up.

7. (From Pure Mathematics 30 January 2001 Diploma Examination, Numerical-Response Question 1)

The graph of $y = f(x)$, where $f(x) = |x + 4| + 1$, is reflected in the y-axis. This produces the same results as would translating the graph of $y = f(x)$ to the right by _____ units.

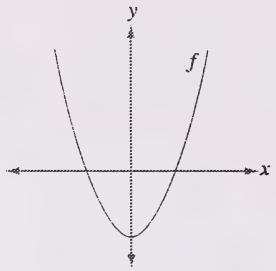
Solution

8

8. (From Pure Mathematics 30 January 2001 Diploma Examination, Multiple-Choice 22)

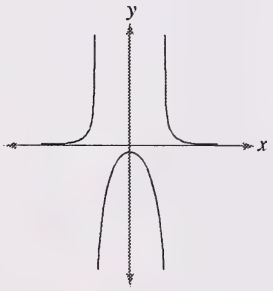
Use the following information to answer the next question.

The partial graph of $y = f(x)$ is shown below.

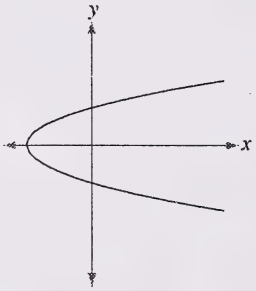


The partial graph of $y = \frac{1}{f(x)}$ is

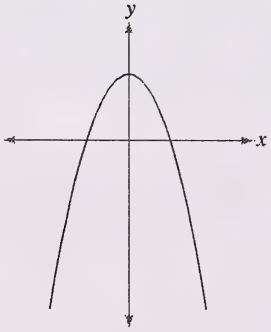
* **A.**



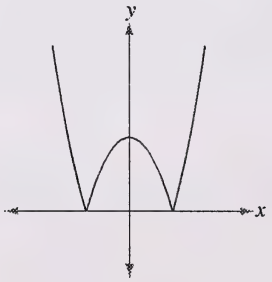
B.



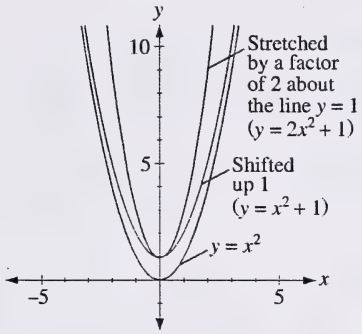
C.



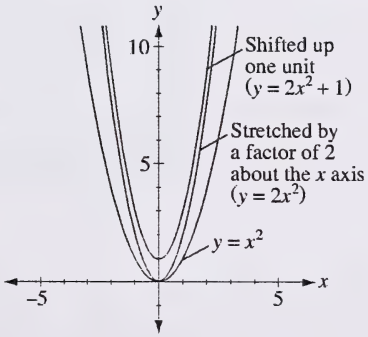
D.



9. The following example shows that order is not important when describing the transformation on the graph $y = 2x^2 + 1$.



Note: Point (2, 4) became point (2, 5) after a translation of 1 unit up. Then, point (2, 5), which is 4 units above $y = 1$, is stretched by a factor of 2 about the line $y = 1$, which is now 8 units above the line $y = 1$, and the point is now (2, 9).



Note: Point (2, 4) became point (2, 8) after being stretched by a factor of 2 about the x-axis, and then it became point (2, 9) with the translation of 1 unit up.

Exponential and Logarithmic Functions and the Geometric Series

General Outcomes

Generate and analyze exponential patterns.

Solve exponential and logarithmic equations and verify identities.

Represent and analyze exponential and logarithmic functions, using technology where appropriate.

General Notes:
- To solve compound interest problems, $A = P(1 + i)^n$ should be used as an example of a geometric sequence.
- Sigma notation should be introduced to denote the sum of a series.
- P6-4 to P6-11 are interrelated. The order of the outcomes may be changed from the *Program of Studies*.
- The window format for graphing calculators is x: $[x_{min}, x_{max}, x_{scl}]$ y: $[y_{min}, y_{max}, y_{scl}]$.

Specific Outcomes

Specific Outcome P6-1

Derive and apply expressions to represent general terms and sums for geometric growth and to solve problems. [CN, R, T]

P6-1 Notes:
- Revisit geometric sequences from Pure Mathematics 10. The general term was not an outcome in Pure Mathematics 10 and should be derived in class. Students are expected to apply examples and problems that incorporate the general term.
- This outcome may be covered as an application of exponential functions.
- Sigma notation to represent geometric series is part of this outcome.
- This outcome is not intended to include recursive definition.

(See examples 1, 2, 3, 5a, 5b, 13, and 20)

Specific Outcome P6-2

Connect geometric sequences to exponential functions over the set of natural numbers. [E, R, V]

P6-2 Notes:

- The use of discrete data and discrete graphs are the focus of this outcome. The solutions are over the set of natural numbers. The use of tables and elements from the graph can be incorporated with technology to solve these problems.
- The exponent is to be a natural number.
- Teachers may wish to graph discrete data or to have students collect real-world data and use exponential regression to find an equation that models their data.

(See example 4)

Specific Outcome P6-3

Estimate values of expressions for infinite geometric processes. [PS, R, T]

P6-3 Notes:

- This outcome is intended as a discovery of number patterns, not as a study of limits.
- Students should have an intuitive understanding of why the sum formula

$S_n = \frac{a(1-r^n)}{1-r}$ becomes $S = \frac{a}{1-r}$ for $|r| < 1$, when n approaches infinity.

(See example 5c)

Specific Outcome P6-4

Solve exponential equations that have numerical bases that are powers of one another. [E, R]

(See example 6)

Specific Outcome P6-5

Solve and verify exponential and logarithmic equations and identities. [R]

P6-5 Notes:
- The emphasis in this outcome is on algebraic methods of solving.
- Exponential and logarithmic equations can be solved on some graphing calculators.
- The proving of exponential and logarithmic identies is not intended.

(See examples 7, 9, 10, and 19)

Specific Outcome P6-6

Graph and analyze an exponential function using technology.

P6-6 Notes:
- Analysis should include domain, range, x- and y-intercepts, and horizontal asymptotes.
- The relationship between the graph of the exponential function and the graph of its inverse (logarithmic function) should be introduced here.
- Graphing and analyzing exponential functions should be done both with and without technology.

(See example 11)

Specific Outcome P6-7

Model, graph, and apply exponential functions to solve problems. [R, T, V]

P6-7 Notes:
- Care must be taken in solving exponential equations because students may need to have previous knowledge of logarithmic laws.
- Students may use exponential regression to find the equation of a curve. The use of exponential regression from a graphing calculator will not result in half-life formulas in the form of $N_t = N_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$, but in the form of $y = ab^x$, where the half-life would have to be calculated.

(See examples 12 and 15)

Specific Outcome P6-8

Change exponential function form to logarithmic function form and vice versa. [CN, E]

P6-8 Note:
- Students must be familiar with the laws of exponents.

(See examples 16, 17, and 18)

Specific Outcome P6-9

Use logarithms to model practical problems. [CN, PS, V]

(See example 8)

Specific Outcome P6-10

Explain the relationship between the laws of logarithms and the laws of exponents. [C, T]

P6-10 Note:
- The laws of logarithms should be used in solving equations and problems.

Specific Outcome P6-11

Graph and analyze logarithmic functions with and without technology. [R, T, V]

P6-11 Note:
- Use technology to explore the shape of an exponential function graph. From it, sketch the graph of the inverse (logarithmic) function without technology.

(See example 14)

Acceptable Standard	*Standard of Excellence*
The student can	The student can also
• generate the terms of a geometric sequence • find the general term of a geometric sequence • write the terms of geometric series given sigma notation • find the sum of any finite geometric sequence • find the sum of a geometric series given the terms or given a general term • solve for a specified term of a geometric sequence and solve for an exponent in an exponential equation by considering tables, graphs, and common bases • discover infinite geometric series through patterning • use a graphing calculator or computer to verify solutions • solve exponential equations in which the bases are rational powers of one another • given a variety of problems, solve by using exponential and/or logarithmic methods • predict the effect on the graph of an exponential function that has had its base value altered • identify domain, range, x-intercepts, y-intercepts, and asymptotes from the graph and from the equation of an exponential function • graph an exponential function by using technology • show the relationship between the graph of an exponential function and the graph of its inverse, which is a reflection in the line $y = x$ • understand the relationship between an exponential function and its inverse, and determine what properties of the function have changed by taking the inverse	• use sigma notation to represent the sum of a geometric sequence • solve for the number of terms, given the sum and other necessary information

Acceptable Standard	*Standard of Excellence*
The student can	The student can also
• find solutions to exponential equation problems when formulas are given	• find solutions to exponential function problems when formulas are not given and must be created algebraically
• use regression methods on a calculator to determine an exponential equation	
• change forms between $y = b^x$ and $\log_b y = x$	• change forms between $y = ab^x$ and $\log_b \frac{y}{a} = x$
• compare values, such as earthquake intensity, given the equation and exponents	• solve for a value or exponent in comparison problems
• explain the relationship between laws of logarithms and laws of exponents	
• use appropriate laws to simplify exponential and logarithmic expressions, and to verify identities	
• identify domain, range, x- and y-intercepts, and asymptote from the graph of a logarithmic function $y = \log_b x$	
• graph logarithmic functions with a base other than 10 by using technology	
• solve exponential and logarithmic equations that involve linear and quadratic equations but may not recognize extraneous solutions	• recognize extraneous solutions when solving exponential or logarithmic equations
• participate in and contribute toward the problem-solving process for problems that can be represented by logarithms or exponential functions studied in Pure Mathematics 30	• complete the solution to problems that can be represented by logarithmic or exponential functions studied in Pure Mathematics 30

Examples

Students who achieve the *Acceptable Standard* should be able to answer all the following questions, except for any part of a question labelled [SE]. Parts labelled [SE] are appropriate examples for students who achieve the *Standard of Excellence*.

1. What is the seventh term in the geometric series $300 + 60 + 12 + \ldots$?

Solution

$$\begin{aligned} t_7 &= 300\left(\frac{1}{5}\right)^{7-1} \\ &= \frac{12}{625} \end{aligned}$$

2. What is the sum of the first 10 terms of the geometric sequence $1 + 3 + 9 + 27 + 81 + \ldots$?

Solution

$$S_{10} = \frac{1(3^{10} - 1)}{3 - 1} = 29\,524$$

3. What is sum of the geometric sequence $\frac{1}{4} + \frac{1}{2} + 1 + 2 + 4 + \ldots 1024$?

Solution

$$\begin{aligned} a &= \frac{1}{4},\ r = 2 \\ 1\,024 &= \frac{1}{4}\left(2^{n-1}\right) \\ 4\,096 &= 2^{n-1} \\ \log 4\,906 &= (n-1)\log 2 \\ 12 &= n - 1 \\ 13 &= n \end{aligned}$$

$$\begin{aligned} S_{13} &= \frac{\frac{1}{4}\left(2^{13} - 1\right)}{2 - 1} \\ &= \frac{8191}{4} \end{aligned}$$

4. The value of an investment is given by $f(x) = 237.50(1.052)^x$, where x is the number of 6-month periods of investment. What is the number of **complete** periods needed in order for the investment to be worth at least \$600.00?

Solution

Use the table function on the calculator or computer spreadsheets to find that
$f(18) = \$591.49$ and $f(19) = \$622.25$
Since $f(18)$ is less than \$600.00 and $f(19)$ is more than \$600.00, $x = 19$ complete periods (9.5 years) are needed for the investment to be worth at least \$600.00.

Alternate Solution

$$\begin{aligned} 600 &= 237.50(1.052)^x \\ 2.5263 &= (1.052)^x \\ \log 2.5263 &= x \log 1.052 \\ \frac{\log 2.5263}{\log 1.052} &= x \\ x &= 18.28 \end{aligned}$$

Therefore, 18 periods are not enough, so 19 periods would be needed.

5. For the geometric series $100 + 50 + 25 + \ldots$,

a. find the sum for 12^{th} term and the sum for the 15^{th} term, both to 4 decimal places

b. explain why these sums are almost equal

c. give the sum for the infinite series

Solution

a. $$S_{12} = \frac{100\left(\left(\frac{1}{2}\right)^{12} - 1\right)}{\frac{1}{2} - 1} = -200\left(\frac{1}{4096} - 1\right) = 199.9512$$

$$S_{15} = \frac{100\left(\left(\frac{1}{2}\right)^{15} - 1\right)}{\frac{1}{2} - 1} = -200\left(\frac{1}{32768} - 1\right) = 199.9939$$

b. These sums are almost equal because $S_{15} = S_{12} + t_{13} + t_{14} + t_{15}$ and t_{13}, t_{14}, t_{15} are so small that they do not change the sum significantly.

c. $$S = \frac{100}{1 - \frac{1}{2}} = 200$$

6. Solve for x in the equation $\left(\frac{1}{8}\right)^{x-3} = 2 \times 16^{2x+1}$.

Solution

$$\begin{aligned}
(2^{-3})^{x-3} &= 2^1 \times (2^4)^{2x+1} \\
2^{-3x+9} &= 2^1 \times (2)^{8x+4} \\
2^{-3x+9} &= 2^{8x+5} \\
\therefore -3x + 9 &= 8x + 5 \\
-11x &= -4 \\
x &= \frac{4}{11}
\end{aligned}$$

7. A radioactive sample decreases to 7.3 mg in 36 h. If the half-life of the radioactive element is 2.8 h, what was the original amount of the radioactive sample, to the nearest milligram?

Solution

$$A = A_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$$

$$7.3 = A_0\left(\frac{1}{2}\right)^{\frac{36}{2.8}}$$

$$\frac{7.3}{\left(\frac{1}{2}\right)^{\frac{36}{2.8}}} = A_0$$

$$54\ 164 \text{ mg} = A_0$$

8. On the Richter scale, the magnitude of an earthquake is given by $m = \log\left(\frac{I}{I_0}\right)$, where I is the earthquake intensity and I_0 is the reference intensity. How many times as intense is an earthquake of magnitude 8.1 than is an earthquake of magnitude 6.4?

Solution

$$m = \log\left(\frac{I}{I_0}\right)$$

$$8.1 = \log\left(\frac{I_1}{I_0}\right)$$

$$-\ 6.4 = \log\left(\frac{I_2}{I_0}\right)$$

$$1.7 = \log\left(\frac{I_1}{I_0}\right) - \log\left(\frac{I_2}{I_0}\right)$$

$$1.7 = \log\left(\frac{I_1}{I_0} \times \frac{I_0}{I_2}\right)$$

$$1.7 = \log\left(\frac{I_1}{I_2}\right)$$

$$\therefore\ 10^{1.7} = \frac{I_1}{I_2}$$

$$50.1 = \frac{I_1}{I_2}$$

$\therefore$ An earthquake of magnitude 8.1 is about 50.1 times as intense as earthquake of magnitude 6.4.

9. Verify that $\left(3^{\log x}\right)\left(3^{\log x}\right) = 3^{\log x^2}$, $x > 0$.

Solution

If $x = 2$,

Left side: $\left(3^{\log 2}\right)\left(3^{\log 2}\right) = 3^{\log 2 + \log 2}$

$= 3^{\log 4}$

Right side: $3^{\log 2^2} = 3^{\log 4}$

$\therefore$ Left Side = Right Side

10. Simplify $\left(3^{\log_5 2}\right)\left(3^{\log_5 4}\right)$.

Solution

$$\left(3^{\log_5 2}\right)\left(3^{\log_5 4}\right) = 3^{(\log_5 2 + \log_5 4)}$$

$$= 3^{\log_5 (2\times 4)}$$

$$= 3^{\log_5 8}$$

11. Use a graphing method to determine the value of x if $3^x = 4$.

Solution

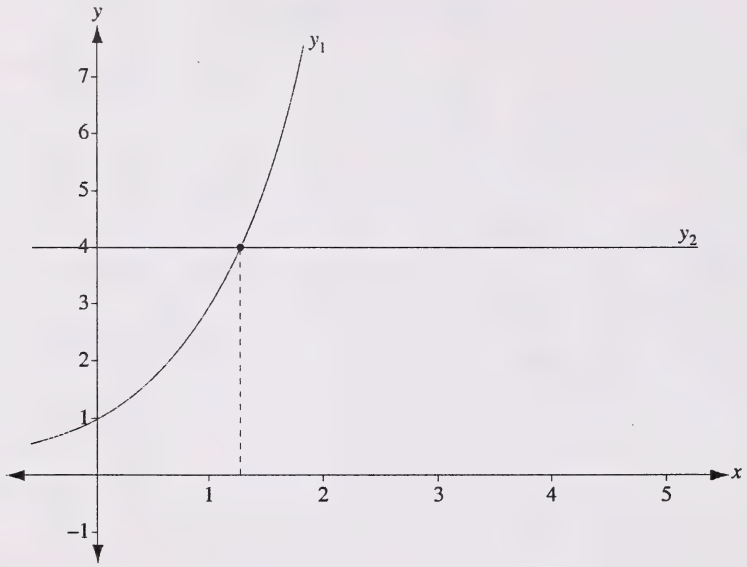
Use window x: [0, 5, 1] y: [–1, 7, 1]

Graph $y_1 = 3^x$.

Graph $y_2 = 4$.

Determine the x value at the intersection point.

The value of x is approximately 1.26.



Alternate Solution

Graph $y = 3^x - 4$ and trace to find the root.

or

Graph $y = 3^x$ and $y = 4$, then calculate the intersection point.

12. The following data represent the cooling of a cup of hot chocolate over time. Use exponential regression to find an equation of the type $T = a(b)^t$, where t = time (min) and T = temperature (°C).

Time (t)	Temperature (T)
0	60
5	54
10	48
15	44
20	41

Solution

Enter the time data in LIST 1 and temperature data in LIST 2 on a graphing calculator. Perform the exponential regression function on the calculator.

This results in the equation $T \doteq 59.4(0.98)^t$.

13. Write the first four terms in the series defined as $\sum_{n=1}^{\infty}[(-1)^{n-1}(2)^{-2n+5}]$.

Solution

$$8, -2, \frac{1}{2}, -\frac{1}{8}$$

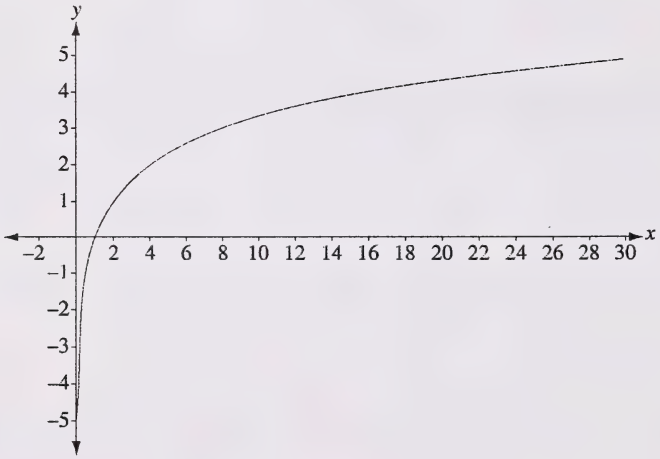
14. Graph $y = \log_2 x$, and identify the domain, range, x- and y-intercepts, and asymptotes.

Solution

$$y = \log_2 x$$
$$2^y = x$$

x	2	1	4	8	$\frac{1}{2}$
y	1	0	2	3	-1

Domain: $x > 0$
Range: $y \in R$
Intercepts x: $(1, 0)$
y: none
Asymptote: $x = 0$ or y-axis



SE **15.** Fred invests \$5 000.00 at an annual interest rate of 6%. When the investment has tripled, Fred will remove the money. After how many full years will Fred be able to remove the money?

Solution

$$\begin{aligned} 15\,000 &= 5\,000(1.06)^n \\ \log 3 &= n \log 1.06 \\ n &= \frac{\log 3}{\log 1.06} \\ &= 18.85 \\ &\text{19 full years} \end{aligned}$$

SE **16.** Given that $y = \left(\frac{1}{2}\right)^{\frac{x}{1.4}}$, show that $x = 1.4 \log_{0.5} y$.

Solution

$$\begin{aligned} \log_{0.5} y &= \frac{x}{1.4} \\ 1.4 \log_{0.5} y &= x \end{aligned}$$

17. (From Pure Mathematics 30 January 2001 Diploma Examination, Multiple-Choice Question 8)

An investment of \$1 000 is earning 4% interest per annum compounded annually. If the value, V, of the investment after t years is given by $V = 1\,000(1.04)^t$, then t, written as a function of V, is

A. $t = \frac{\log(V)}{3} - \log(1.04)$

B. $t = \frac{\log(V)}{3 \log(1.04)}$

C. $t = \log(V) - 3 - \log(1.04)$

* **D.** $t = \frac{\log(V) - 3}{\log(1.04)}$

SE **18.** Given $\log_3 a = 1.43$ and $\log_4 b = 1.86$, determine $\log_b a$.

Solution

$$3^{1.43} = a \qquad \therefore 1.43 \log 3 = \log a$$

$$4^{1.86} = b \qquad \therefore 1.86 \log 4 = \log b$$

$$\log_b a = \frac{\log a}{\log b}$$

$$\frac{1.43 \log 3}{1.86 \log 4} \doteq 0.609$$

19. (From Pure Mathematics 30 January 2001 Diploma Examination, Multiple-Choice Question 6)

If $\log_3 y = c - \log_3 x$, where $y > 0$ and $x > 0$, then y is equal to

A. $c - x$

B. $\frac{c}{x}$

C. $\frac{c^3}{x}$

* **D.** $\frac{3^c}{x}$

SE **20.** Earthquake intensity is given by $I = I_o(10)^m$, where I_o is the reference intensity and m is magnitude. A particular major earthquake of magnitude 7.9 is 120 times as intense as a particular minor earthquake. What is the magnitude of the minor earthquake?

Solution

$$\begin{aligned} I_1 &= I_o(10)^{7.9} \\ \div\ I_2 &= I_o(10)^m \\ \hline \frac{I_1}{I_2} &= (10)^{7.9-m} \\ 120 &= (10)^{7.9-m} \\ \log 120 &= \log 10^{7.9-m} \\ \log 120 &= (7.9 - m)\log 10 \\ m &= 7.9 - \log 120 \\ m &= 5.8 \end{aligned}$$

SE **21.** Solve $\log_7(x+1) + \log_7(x-5) = 1$, and verify your solution.

Solution

$$\log_7(x+1) + \log_7(x-5) = 1$$
$$\log_7[(x+1)(x-5)] = 1$$
$$7^1 = x^2 - 4x - 5$$
$$0 = x^2 - 4x - 12$$
$$0 = (x-6)(x+2)$$
$$x-6 = 0 \quad \text{or} \quad x+2 = 0$$
$$x = 6 \quad \text{or} \quad \cancel{x=-2} \quad \text{extraneous root}$$

Verification:

$$\log_7(6+1) + \log_7(6-5) = 1$$
$$\log_7 7 + \log_7 1 = 1$$
$$1 + 0 = 1$$

$$\log_7(-2+1) + \log_7(-2-5) = 1$$
$$\underbrace{\log_7(-1) + \log_7(-7)} = 1$$

impossible $\therefore x = -2$ is an extraneous root

SE **22.** Write the geometric series $3 - 6 + 12 - 24 + \ldots + 192$ in summation notation.

Solution

$r = -2$

$a = 3$

$$192 = 3(-2)^{n-1}$$
$$64 = (-2)^{n-1}$$
$$(-2)^6 = (-2)^{n-1}$$
$$6 = n-1$$
$$7 = n$$

$$\sum_{N=1}^{7} 3(-2)^{N-1}$$

General Outcomes

Solve trigonometric equations and identities

Represent and analyze trigonometric functions, using technology where appropriate

General Notes:
- The window format for graphing calculators is x: $[x_{min}, x_{max}, x_{scl}]$ y: $[y_{min}, y_{max}, y_{scl}]$.

Specific Outcomes

Specific Outcome P8-1

Distinguish between degree and radian measure, and solve problems using both. [C, CN, E]

Represent and analyze trigonometric functions, using technology where appropriate.

P8-1 Notes:
- Students need to be aware that their calculators must be in the correct angle mode.
- When cleared, calculators default to **radian** mode.
- Students must carry the accuracy of their calculators forward through all parts of the question. **Rounding must not occur until the final solution is determined.**
- Solving problems involving arc length, radius, and angle measure in either radian or degrees should be included.

(See examples 1, 2, 3, 4, 5, and 9)

Specific Outcome P8-2

Determine the exact and the approximate values of the six trigonometric ratios for any integral multiples of 0°, 30°, 45°, 60°, and 90°, and for 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, and $\frac{\pi}{2}$. [CN, E]

P8-2 Notes:

- The reciprocal ratios should be introduced at this time.
- These questions can be done as exact values on graphing calculators that have symbolic manipulation capabilities. They will not be assessed as individual questions on the diploma examination, but as parts of more involved questions.
- Special triangles are not dealt within the specific outcomes of the Pure Mathematics 10 or 20 programs.

(See examples 6 and 7)

Specific Outcome P8-3

Solve first- and second-degree trigonometric equations over a domain of length 2π

- algebraically
- graphically

[PS, T]

P8-3 Notes:

- Students with graphing calculators that have symbolic manipulation capabilities can solve these problems through the "solve" feature, either in approximate or exact form. On the diploma examination, these questions will be a part of a more involved question.
- Restrict reciprocal function equations to first degree only.
- Solutions to equations can only be approximated from the graph.

(See examples 8, 10, 11, and 13)

Specific Outcome P8-4

Determine the general solutions to trigonometric equations where the domain is the set of real numbers. [PS, T]

P8-4 Notes:

- The student should be able to write an expression for the general solution to a trigonometric equation.
- The student should be able to solve trigonometric equations over the domain of the real numbers.
- Some of the examples or questions could be done with graphing calculators that have symbolic manipulation capabilities by entering statements such as solve $(x - 2\sin x = 0, x)$

(See example 12)

Specific Outcome P8-5

Verify trigonometric identities
- numerically for any particular case
- algebraically for general cases (i.e., proving identities)
- graphically

[PS, R, T, V]

P8-5 Note:
- Trigonometric functions may have either point or asymptote discontinuities that must be addressed in the analysis of the functions.

(See examples 14, 15, and 27)

Specific Outcome P8-6

Use sum, difference, and double-angle identities for sine and cosine to verify and simplify trigonometric expressions. [R, T]

P8-6 Notes:
- Students should be able to use the sum identities to prove, derive, and simplify the double-angle identities of $\sin 2x$ and $\cos 2x$.
- Students should be able to express the $\cos 2x$ identity in three different ways.

(See examples 16, 17, and 26)

Specific Outcome P8-7

Describe the three primary trigonometric functions as circular functions with reference to the unit circle and an angle in standard position. [PS, R, V]

P8-7 Notes:
- This outcome should also be understood for the three reciprocal ratios.
- The trigonometric functions of θ should be defined in terms of x, y, and r for all points on the coordinate plane (including quadrants III and IV).
- This concept may be developed by using the unit circle, special triangles, symmetry, etc.

(See examples 19 and 20)

Specific Outcome P8-8

Graph (using technology), sketch, and analyze the graphs of sine, cosine, and tangent functions, for

- amplitude, if defined
- period
- domain and range
- asymptotes, if any
- behaviour under transformations

P8-8 Notes:

- Behaviour of functions under transformations includes horizontal phase shift and vertical displacement.
- Transformations that are stretches and compressions can be described by using terms such as amplitude and period.
- Discussion of all four parameters a, b, c, and d should only be done for sinusoidal functions.
- Sinusoidal functions will be expressed as $y = a \sin[b(x - c)] + d$ or $y = a \cos[b(x - c)] + d$, where $a > 0$. Students using regression models on graphing calculators have the responsibility of adjusting to this form.
- Remember that the absolute value of basic trigonometric functions should be addressed. (See outcome P9-8)
- It is not intended that inverse trigonometric functions (e.g., $y = \sin^{-1}x$ or $y = \arcsin x$) be studied in this course. They are to be used only in calculations.

(See examples 18, 21, 22, 23, and 24)

Specific Outcome P8-9

Graph (using technology), sketch, and analyze the graphs of secant, cosecant, and cotangent functions for

- period
- domain and range
- asymptotes
- behaviour under transformations

P8-9 Notes:

- If $f(x) = \sin x$, the reciprocal function is $\csc x$ and must not be denoted by $\sin^{-1}x$.
  $\left(\text{i.e., } \frac{1}{f(x)} \text{ is not the same as } f^{-1}(x)\right)$
- Only behaviour under single parameter transformations should be discussed.
- The importance of selecting the correct window on a graphing calculator should be discussed.

(See example 28)

Specific Outcome P8-10

Use trigonometric functions to model and solve problems. [PS, R,V, T]

P8-10 Notes:

- In modelling problems based on trigonometric functions, students should be able to change the horizontal axis from angle measurement to time units.
- In solving for t in practical problems, students should use technology as opposed to an algebraic method.

(See examples 25 and 29)

Acceptable Standard	*Standard of Excellence*
The student can	The student can also
• convert from degrees to radians and vice versa • solve problems involving arc length, radius, and angle measure in either radians or degrees	
• find exact values of trigonometric ratios for special angles, θ where $0 \leq \theta < 2\pi$ or $0° \leq \theta < 360°$	• find exact values of trigonometric ratios for any integral multiples of the special angles
• explain why certain trigonometric ratios can have undefined values	
• identify restrictions on the variable in the domain $0 \leq \theta < 2\pi$	• identify restrictions on the variable in the domain $\theta \in R$
• algebraically and graphically determine – all solutions to first-degree equations – partial solutions to second-degree equations – partial solutions to trignometric equations involving reciprocal trigonometric ratios – partial solutions to trigonometric equations involving multiple angles	• algebraically and graphically determine – all solutions to first- and second-degree equations – the general solution of trigonometric equations – all solutions to trigonometric equations involving reciprocal trigonometric ratios – all solutions to trigonometric equations involving multiple angles
• convert reciprocal trigonometric ratios to primary ratios in order to to solve equations	
• verify identities for a particular case	
• simplify and prove simple identities algebraically for a general case	• simplify and prove more difficult identities algebraically for the general case, such as those requiring use of conjugates or extensive use of rational operations
• graph both sides of an identity to verify it	
• verify double-angle identities of sine and cosine for a particular case	• algebraically prove identities involving double-angle identities
• simplify expressions by using double-angle identities of sine and cosine	
• graph expressions involving double-angles	• analyze graphs of expressions involving double angles
• find exact values for the six trigonometric ratios, given a point on the terminal arm of the angle	• find other trigonometric ratios given one ratio and a quadrant specification
• find an angle such that its terminal arm passes through a given point	

Acceptable Standard	*Standard of Excellence*
• create a graph using technology for primary trigonometric functions and analyze the graph for features such as domain, range, amplitude, period, and asymptotes	• create a graph using technology for reciprocal trigonometric functions and analyze the graph for features such as domain, range, amplitude, period, and asymptotes
• give partial explanations of the relationships beween equation parameters and transformations of sinusoidal waves	• give full explanations of the relationships between equation parameters and transformations of sinusoidal waves
• perform single transformations of reciprocal trigonometric functions	
• graph a sinusoidal wave to model a problem, given the equation	
• develop the partial equation for a sinusoidal wave, given the graph	• develop the complete equation of a sinusoidal wave, given a description of the problem or the graph
• answer questions about a problem, given the equation or the graph	• create a graph of a sinusoidal wave, given a description of the problem
• participate in and contribute toward the problem-solving process for problems that require the analysis of trigonometry studied in Pure Mathematics 30	• complete the solution to problems that require the analysis of trigonometry studied in Pure Mathematics 30

Examples

Students who achieve the *Acceptable Standard* should be able to answer all the following questions, except for any part of a question labelled **SE**. Parts labelled **SE** are appropriate examples for students who achieve the *Standard of Excellence*.

1. Convert the angles $\frac{2\pi}{3}$ and 1.6 rad to degrees, correct to the nearest tenth.

Solution

$\frac{2\pi}{3} = 120.0°$

1.6 rad = 91.7°

2. **a.** Express 240° in radians, in terms of π.

b. Express 218° in radians, correct to the nearest hundredth.

Solution

a. $240° = \frac{4\pi}{3}$

b. 218° = 3.80 rad

3. In an experiment to verify the Law of Refraction, a student took measurements (in degrees) of the angle of incidence and the angle of refraction. The student used a standard spreadsheet program to calculate the sine of each angle and the ratio of the two sine values. The results of the spreadsheet calculation are shown in the following table. The student knew that the ratio of $\frac{\sin i}{\sin r}$ was supposed to be approximately 1.5.

Angle of Incidence i (degrees)	Angle of Refraction r (degrees)	$\sin i$	$\sin r$	$(\sin i)/(\sin r)$
10	7	−0.544	0.657	−0.83
20	13	0.913	0.420	2.17
30	19	−0.988	0.150	−6.59
40	25	0.745	−0.132	−5.63
50	30	−0.262	−0.988	0.27
60	35	−0.305	−0.428	0.71
70	38	0.774	0.296	2.61
80	40	−0.994	0.745	−1.33

a. Examine the spreadsheet and explain how it shows that the calculations are not what the student expected to find.
b. In the calculations of $\sin i$ and $\sin r$, does the spreadsheet show work in degree mode or radian mode?
c. Explain what changes would have to be made to modify the spreadsheet so that all entries would reflect radian measure.
d. Modify the spreadsheet so that the correct sine values are calculated.

Solution

a. The student expected the ratio to be close to 1.5; however, it is not.
b. Radian mode
c. Column 1 and 2 entries should be multiplied by $\frac{\pi}{180}$.
d.

Angle of Incidence i (radians)	Angle of Refraction r (radians)	$\sin i$	$\sin r$	$(\sin i)/(\sin r)$
0.175	0.122	0.174	0.122	1.43
0.349	0.227	0.342	0.225	1.52
0.524	0.332	0.500	0.326	1.54
0.698	0.436	0.643	0.423	1.52
0.873	0.534	0.766	0.500	1.53
⋮	⋮	⋮	⋮	⋮

4. A student constructs an arc of a circle by rotating a radius of 5 cm through an angle of 250°.
   a. Convert the angle to radian measure, correct to the nearest tenth of a radian.
   b. Find the length, correct to the nearest tenth of a centimetre, of the arc subtended by this angle.
   c. Explain how the angle measurement, in radians, relates the arc length to the radius of the circle.

Solution

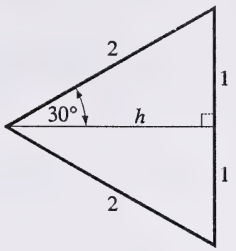
   a. angle measure = 4.4 rad
   b. arc length = 21.8 cm
   c. An angle of 4.4 rad means that the arc length is 4.4 times the length of the radius of the circle.

5. a. Given that $\sin\theta = 0.5395$ and that θ is an acute angle, what is θ in radians, to the nearest hundredth?
   b. Given that $\sin\theta = 0.5395$ and that $0 \le \theta < 2\pi$, are there any additional solutions? If so, find them and explain why they are also solutions.
   c. Explain why $\sin\theta = 1.5395$ has no solution for θ.

Solution

   a. $\theta = 0.57$ radians
   b. Yes, since the sine ratio is also positive in quadrant II.
      $\theta = \pi - 0.57 = 2.57$ radians.
   c. Because $\sin\theta = \frac{y}{r}$, and $y \le r$, therefore $\frac{y}{r}$ has to be ≤ 1.

6. Given an equilateral triangle with a side of 2 units, determine the exact trigonometric ratios of 30°.



Solution

$$2^2 = 1^2 + h^2$$
$$3 = h^2$$
$$\sqrt{3} = h$$

$$\cos 30° = \frac{\sqrt{3}}{2},\ \sin 30° = \frac{1}{2},\ \tan 30° = \frac{1}{\sqrt{3}} \text{ or } \frac{\sqrt{3}}{3}$$

7. **a.** Find the exact values for $\sin\frac{7\pi}{6}$, $\tan\frac{2\pi}{3}$, and $\sec\frac{7\pi}{4}$.

b. Find the exact values for $\tan\left(-\frac{2\pi}{3}\right)$ and $\csc\left(-\frac{8\pi}{3}\right)$.

Solution

a. $\sin\frac{7\pi}{6} = -\frac{1}{2}$, $\tan\frac{2\pi}{3} = -\sqrt{3}$, $\sec\frac{7\pi}{4} = \sqrt{2}$

b. $\tan\left(-\frac{2\pi}{3}\right) = \sqrt{3}$, $\csc\left(-\frac{8\pi}{3}\right) = -\frac{2}{\sqrt{3}}$ or $-\frac{2\sqrt{3}}{3}$

8. Algebraically and graphically find the solutions to the following trigonometric equations.

a. $1 + 2\cos x = 5\cos x$, where $0 \le x < 2\pi$ Give solutions in decimal form, correct to the nearest hundredth.

SE **b.** $\cos 4x = 0.5$, where $0 \le x \le 2\pi$ Give solutions as exact values.

Solution

a.
$$1 = 3\cos x$$
$$\frac{1}{3} = \cos x$$
$$x = 1.23,\ 5.05$$

SE **b.** $\cos 4x = \frac{1}{2}$ where $0 \le x \le 2\pi \quad \therefore 0 \le 4x \le 8\pi$

$$4x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \frac{13\pi}{3}, \frac{17\pi}{3}, \frac{19\pi}{3}, \frac{23\pi}{3}$$
$$x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$$

9. **a.** Explain why $\tan\frac{\pi}{2}$ is undefined.

b. Are there any of the remaining five trigonometric ratios that have undefined values? Explain why.

Solution

a. $\tan A = \frac{y}{x}$ When $x = 0$, $\frac{y}{0}$ is undefined. This occurs at $A = \frac{\pi}{2}$.

b. Yes, $\csc A$, $\sec A$, and $\cot A$ have undefined values because their denominators can also equal zero.

10. Given $3\sec A = 6$, where $0 \le A < 2\pi$, what are the exact values of A?

Solution

$$3\sec A = 6$$
$$\sec A = 2$$
$$\cos A = \frac{1}{2}$$
$$A = \frac{\pi}{3}, \frac{5\pi}{3}$$

11. **a.** Graph $y = 6\sin^2 A - \sin A - 1$, using the window $x: \left[\frac{-\pi}{2}, \frac{5\pi}{2}, \frac{\pi}{2}\right]$, $y: [-2, 6, 1]$ on your graphing calculator.

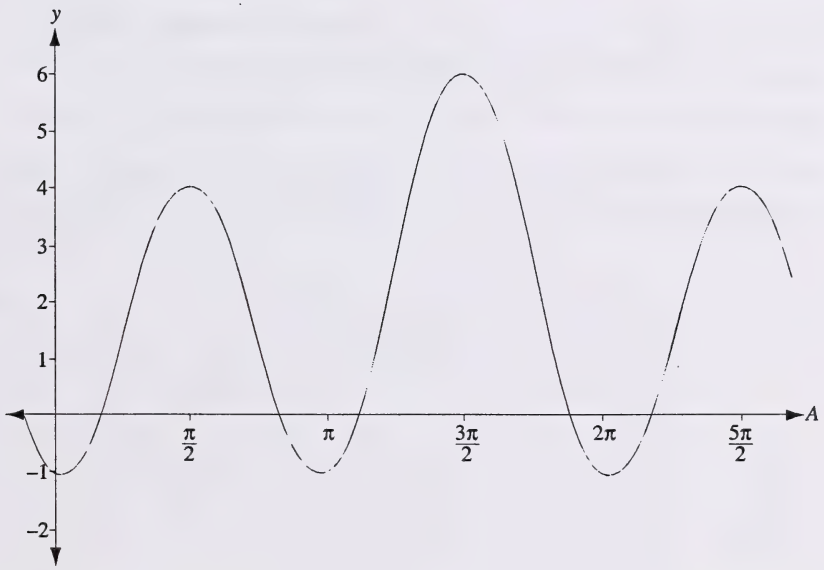
b. Predict the number of solutions to $6\sin^2 A - \sin A - 1 = 0$, $0 \le A \le 2\pi$, from your graph and give a reason for your prediction.

c. Using your calculator, estimate the solutions of $6\sin^2 A - \sin A - 1 = 0$, $0 \le A \le 2\pi$, to the nearest tenth.

SE **d.** Confirm your estimates algebraically by finding A, correct to the nearest hundredth, for $0 \le A < 2\pi$.

Solution

a.



b. Four solutions exist because there are 4 intercepts between 0 and 2π on the horizontal axis.

c. 0.5, 2.6, 3.5, 5.9

SE **d.** $6\sin^2 A - \sin A - 1 = 067$

$(2\sin A - 1)(3\sin A + 1) = 0$

$\sin A = \frac{1}{2}, -\frac{1}{3}$

$A = 0.52, 2.62, 3.48, 5.94$ radians

12. Determine the general solution of $(2\sin A - 1)(\tan A - 1) = 0$.

Solution

$\sin A = \frac{1}{2}$ or $\tan A = 1$

$A = \frac{\pi}{6} + 2n\pi,\ \frac{5\pi}{6} + 2n\pi,\ \frac{\pi}{4} + n\pi,\ n \in I$

SE Student who achieve the *Acceptable Standard* can find partial answers (e.g., only give one solution for one rotation). Student who achieve the *Standard of Excellence* can write an expression for the general solution.

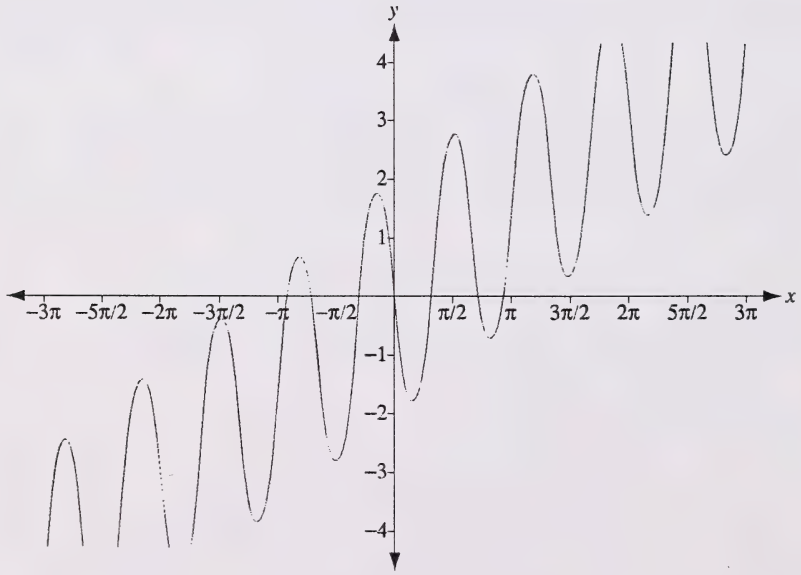
13. Use technology to graph $y = \frac{x}{2} - 2\sin 3x$ in radian mode, and then

a. determine the total number of x-intercepts over the real numbers

b. determine the lowest possible value of x, to the nearest tenth, for $\frac{x}{2} - 2\sin 3x = 0$

c. explain how a student could answer part a by entering $y_1 = \frac{x}{2}$ and $y_2 = 2\sin 3x$ on a graphing calculator

Solution



a. 7

b. −2.9

c. The student could count the number of points of intersection of the two graphs.

14. **a.** Verify $\sin^2 x + \cos^2 x = 1$ for $x = \frac{5\pi}{3}$.

b. Graph $y = \sin^2 x + \cos^2 x$ and describe the result.

c. Explain the result observed in b.

Solution

a. $$\sin^2\left(\frac{5\pi}{3}\right) + \cos^2 \frac{5\pi}{3} = \left(-\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2$$
$$= \frac{3}{4} + \frac{1}{4}$$
$$= 1$$

b. The graph is a horizontal line at $y = 1$.

c. $\sin^2 x + \cos^2 x = 1$ for all values of x, therefore it is an identity.

15. **a.** Prove algebraically that $\sin x \cot x = \cos x$, $\sin x \neq 0$.

b. Given that x is an acute angle, determine x, correct to the nearest degree, such that $\sin x \cot x = 0.4$

Solution

a. $$\sin x \cot x = \sin x \frac{\cos x}{\sin x}, \quad \sin x \neq 0$$
$$= \cos x$$

b. $$\sin x \cot x = \cos x$$
$$\cos x = 0.4$$
$$x = 66°$$

16. Given the identity $\frac{\sin x}{1-\cos x} = \frac{1+\cos x}{\sin x}$, $\cos x \neq 1$ and $\sin x \neq 0$,

a. verify the identity for the particular case $x = \frac{\pi}{3}$

b. demonstrate how the graphs verify the identity, by graphing $y_1 = \frac{\sin x}{1-\cos x}$ and $y_2 = \frac{1+\cos x}{\sin x}$

SE **c.** prove the identity for a general angle by using an algebraic approach

SE **d.** identify the values of x for which this identity is undefined

Solution

a.

Left Side	Right Side
$\frac{\sin\frac{\pi}{3}}{1-\cos\frac{\pi}{3}} = \frac{\frac{\sqrt{3}}{2}}{1-\frac{1}{2}} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$	$\frac{1+\cos\frac{\pi}{3}}{\sin\frac{\pi}{3}} = \frac{1+\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{\frac{3}{2}}{\frac{\sqrt{3}}{2}} = \frac{3}{\sqrt{3}} = \sqrt{3}$

b. The equations $y_1 = \frac{\sin x}{1-\cos x}$ and $y_2 = \frac{1+\cos x}{\sin x}$ produce identical graphs. Thus, $\frac{\sin x}{1-\cos x} = \frac{1+\cos x}{\sin x}$ for all x is a reasonable conjecture.

SE **c.** $\left(\frac{\sin x}{1-\cos x}\right)\left(\frac{1+\cos x}{1+\cos x}\right) = \frac{\sin x(1+\cos x)}{1-\cos^2 x} = \frac{\sin x(1+\cos x)}{\sin^2 x} = \frac{1+\cos x}{\sin x}$

SE **d.** $\frac{\sin x}{1-\cos x}$ is undefined when $\cos x = 1$; therefore, $x \neq 2n\pi$, $n \in I$

$\frac{1+\cos x}{\sin x}$ is undefined when $\sin x = 0$; therefore, $x \neq n\pi$, $n \in I$

$\therefore\ x \neq n\pi$, $n \in I$

17. **a.** Show that $\sin 2x = 2 \sin x \cos x$.

b. Write $2(\sin 5A)(\cos 5A)$ as a single trigonometric expression.

Solution

a. $\sin 2x = \sin(x + x)$

$= \sin x \cos x + \cos x \sin x$

$= 2 \sin x \cos x$

b. $\sin(10A)$

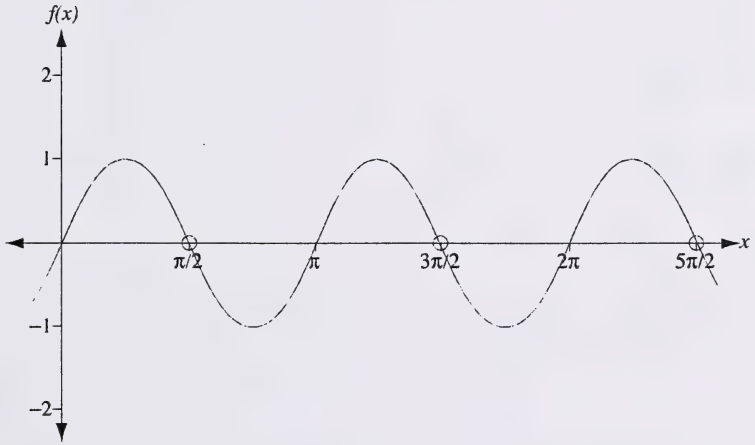
18. Given the function $f(x) = \dfrac{2\tan x}{1+\tan^2 x}$, $x \neq \dfrac{\pi}{2} + n\pi$ and $n \in I$,

a. graph the function and predict the period of $f(x)$

SE **b.** simplify $f(x)$ to a single trigonometric function

Solution

a.



Period = π

Note: $f(x)$ is not continuous, and $x \neq \dfrac{\pi}{2} + n\pi$, $n \in I$

SE **b.** $f(x) = \dfrac{2\tan x}{1+\tan^2 x}$

$$= \frac{\dfrac{2\sin x}{\cos x}}{\sec^2 x}$$

$$= \frac{\dfrac{2\sin x}{\cos x}}{\dfrac{1}{\cos^2 x}}$$

$$= \frac{2\sin x}{\cos x} \times \frac{\cos^2 x}{1}$$

$$= 2\sin x \cos x$$

$$= \sin 2x$$

$x \neq \dfrac{\pi}{2} + n\pi$, $n \in I$

19. The terminal arm of an angle of rotation, θ, passes through the point $(-3, 5)$.

a. Find the exact values of the six trigonometric ratios for angle θ.

b. Determine the smallest positive value of θ, correct to the nearest degree.

Solution

a. $\tan\theta = -\frac{5}{3} \quad \cot\theta = -\frac{3}{5}$

$$r^2 = (-3)^2 + 5^2$$
$$r^2 = 34$$
$$r = \sqrt{34}, \ r > 0$$

$$\sin\theta = \frac{5}{\sqrt{34}} \qquad \csc\theta = \frac{\sqrt{34}}{5}$$

$$\cos\theta = -\frac{3}{\sqrt{34}} \qquad \sec\theta = -\frac{\sqrt{34}}{3}$$

b. The calculator shows $\theta = -59°$, but θ is a second quadrant angle

$\therefore \ \theta = 180° + -59° = 121°$

20. Given that $\cos\theta = -\frac{7}{8}$ and $\tan\theta > 0$, determine the exact value of $\sin\theta$.

Solution

Since $\cos < 0$ and $\tan\theta > 0$, θ is a third quadrant angle.

$$\sin^2\theta + \cos^2\theta = 1$$
$$\sin^2\theta + \left(-\frac{7}{8}\right)^2 = 1$$
$$\sin^2\theta = 1 - \frac{49}{64}$$
$$\sin^2\theta = \frac{15}{64}$$
$$\therefore \ \sin\theta = -\frac{\sqrt{15}}{8}, \text{ because } \theta \text{ is a third quadrant angle}$$

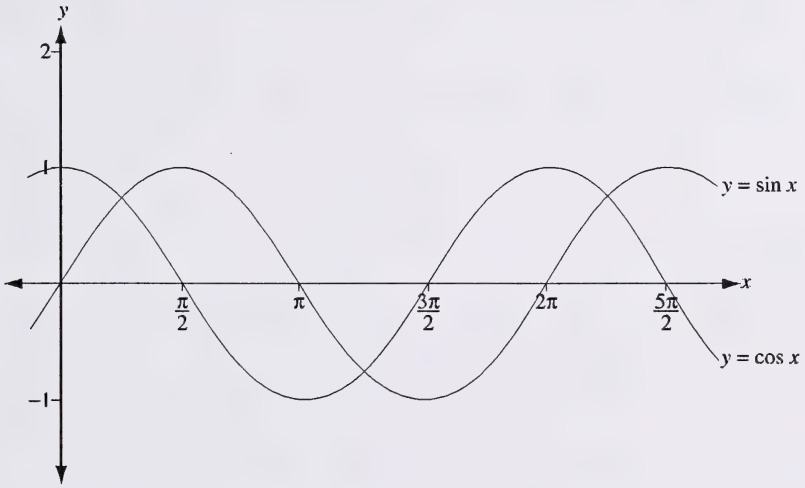
21. Using a graphing utility, graph one cycle of $y = \sin x$ and one cycle $y = \cos x$ on the same set of axes.

a. What relationship exists between the two graphs?

b. What is the amplitude and period of each graph?

Solution

a.

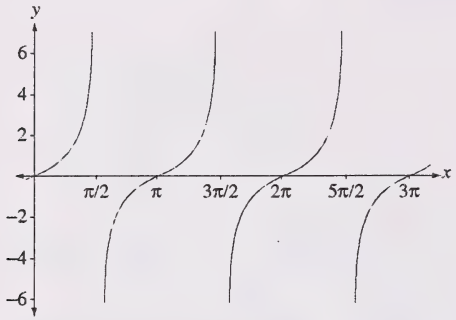


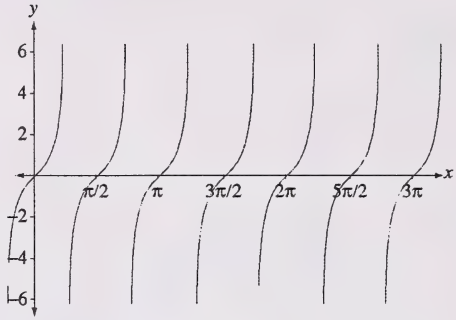
There is a horizontal phase shift of $\frac{\pi}{2}$ rad to the right from $y = \cos x$ to $y = \sin x$.

b. Amplitude = 1
Period = 2π

22. Graph $y = \tan x$ and $y = \tan 2x$. Compare the period, the domain, and the range of $y = \tan x$ with those of $y = \tan 2x$.

Solution





Period changes from π to $\frac{\pi}{2}$.

Domain changes from $x \neq \frac{\pi}{2} \pm n\pi,\ n \in I,$ to $x \neq \frac{\pi}{4} \pm \frac{n\pi}{2},\ n \in I.$

The range, $y \in R,$ remains the same.

23. For the equation $y = a\sin[b(x-c)] + d$, where $a = 4$, $b = 3$, $c = -\frac{3\pi}{4}$, and $d = -3$, determine the

a. domain and range
b. amplitude
c. period
d. x-intercepts, $0 \le x \le \pi$, and y-intercept, correct to the nearest hundredth
e. horizontal phase shift
f. vertical displacement

Solution

a. Domain: $x \in R$
Range: $-7 \le y \le 1$
b. Amplitude: 4
c. Period: $\frac{2\pi}{3}$
d. x- intercepts: 0.02, 0.5, 2.12, 2.60
y-intercept: -0.17
e. left $\frac{3\pi}{4}$
f. down 3 units

24. Compare the graph of $y_2 = 4\sin\left[3\left(x + \frac{3\pi}{4}\right)\right] - 3$ with the graph of $y_1 = \sin x$ with respect to

a. domain
b. amplitude
c. period
d. x-intercepts, $0 \le x \le \pi$, and y-intercepts
e. horizontal phase shift
f. vertical displacement

Solution

a. Domain: $x \in R$, for both graphs
Range: $-7 \le y \le 1$ for y_2, compared with $-1 \le y \le 1$ for y_1

b. Amplitude: 4 for y_2, compared with 1 for y_1

c. Period: $\frac{2\pi}{3}$ for y_2, compared with 2π for y_1

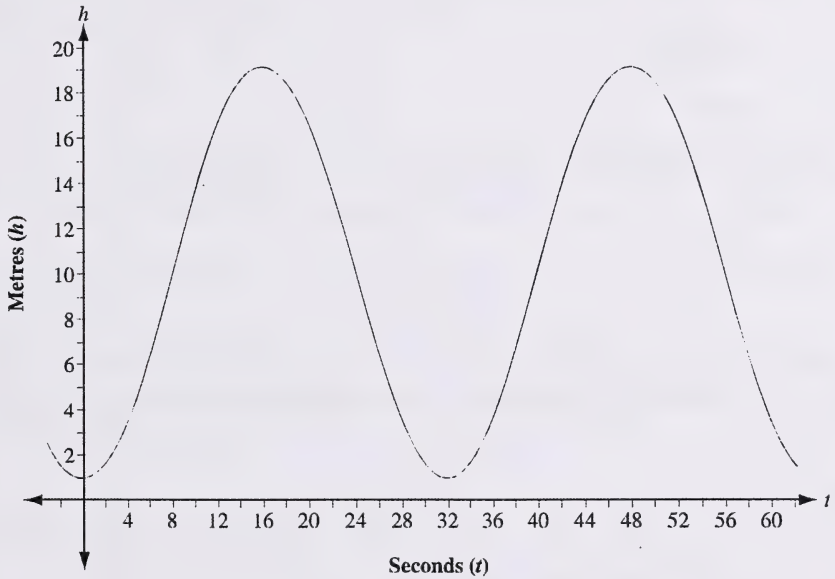
d. x- intercepts: 0.02, 0.5, 2.12, 2.60 for y_2, compared with 0 and 3.14 for y_1
y-intercepts: -0.17 for y_2, compared with 0 for y_1

e. left $\frac{3\pi}{4}$

f. down 3 units

Use the following information to answer the next question.

The graph below shows the height, h, in metres above the ground, over time, t, in seconds that it takes a ferris wheel to make one complete revolution of the ride. The maximum height of the ferris wheel is 19 m and the minimum height is 1 m.



25. **a.** What is the period for 1 revolution of the ferris wheel?

b. How high is the hub or centre of the ferris wheel off the ground?

SE **c.** Write an equation for the height, h, of the ferris wheel as a function of time, t. Use the sine function for h in terms of t.

d. Find the distance from the ground, to the nearest tenth of a metre, of a particular point on the ride at $t = 10$ s.

e. Using technology, find the first time, to the nearest tenth of a second, that a particular point on the ferris wheel is 6 m above the ground.

Solution

a. $P = 32$ seconds

b. Height of hub $= \frac{19+1}{2} = 10$ m high

SE **c.** $h = 9\sin\left[\frac{2\pi}{32}(t-8)\right] + 10$

d. $h = 9\sin\left[\frac{2\pi}{32}(10-8)\right] + 10$

$h = 13.4$ m

e. Let $y_1 = 9\sin\left[\frac{2\pi}{32}(t-8)\right] + 10$

and $y_2 = 6$, and use **INTERSECT** function

$\therefore\ t \doteq 5.7$ s.

SE **26.** Given $\cos 2A = \frac{1}{\sqrt{2}}$, where $0 \le A < 2\pi$, find the exact values of A.

Solution

$$2A = \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}$$

$$A = \frac{\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{15\pi}{8}$$

Note: For double-angle equations, the domain becomes $0 \le 2A < 4\pi$.
Student who achieve the *Acceptable Standard* can find two of the four solutions.

SE **27.** Prove algebraically that $\frac{\sin 2x}{1+\cos 2x} = \tan x$, $x \ne \frac{\pi}{2} + n\pi$, $n \in I$.

Solution

$$\frac{\sin 2x}{1+\cos 2x} = \frac{2\sin x\cos x}{1+(2\cos^2 x - 1)}$$

$$= \frac{2\sin x\cos x}{2\cos^2 x}$$

$$= \frac{\sin x}{\cos x}$$

$$= \tan x$$

SE **28.** Determine the domain, range, and period of $f(x) = \cot x$ and $g(x) = \cot 2x$.

Solution

The domain of $f(x) = \cot x$ is $x \neq n\pi$, $n \in I$, and period is π.

The domain of $g(x) = \cot 2x$ is $x \neq \frac{n\pi}{2}$, $n \in I$, and period is $\frac{\pi}{2}$.

The range is the same for both functions: $y \in R$.

SE **29.** In a 366-day year, the average daily maximum temperature in Vancouver follows a sinusoidal pattern with a highest value of 23.6°C on day 208, July 26, and a lowest value of 4.2°C on day 26, January 26.

a. Describe the variation of temperature, T, with a sine or cosine equation in terms of time in days, d.
b. What is the expected maximum temperature for day 120, May 26?
c. How many days will have an expected maximum temperature of 21.0°C or higher?
d. Explain what changes you would make to a cosine equation so that there would be no phase shift.

Solution

a. $y = 9.7 \sin\left[\frac{2\pi}{366}(x - 117)\right] + 13.9$ approximately

b. $y = 9.7 \sin\left[\frac{2\pi}{366}(120 - 117)\right] + 13.9$

$y = 14.4°$

c. $21 = 9.7 \sin\frac{2\pi}{366}(x - 117) + 13.9$

$x = 164.8$. Day 165 is first day over 21°, and temperature continues to stay over 21° until about day 252.

$\therefore$ Number of days $= 252 - 165 = 87$

d. To avoid a phase shift, the year would start on July 26 instead of Jan 1.

Conics

General Outcome

Classify conic sections according to their shapes and equations.

General Notes:

- The transformations of relations from specific outcome P9-9 (Describe and perform single transformations and combinations of transformations on functions and relations) will be assessed as part of the conics unit on the diploma examination. The terminology relating to this outcome should be common to the conics unit and transformations unit.
- Students will be required to identify the conic formed by the intersection of a plane with a right circular cone. Students will also be required to identify the type of conic formed by a given equation in general or standard form. The intent is that students look at the graph and analyze the transformations, not that they study the relationship between a, b, and c as parameters in the standard equation.
- The use of the graphing calculator and the standard conics program or a graphing utility on a computer may be used as a teaching tool; however, since calculator memories must be cleared on the diploma examination, students cannot rely on this program to determine the type of conic.
- The use of eccentricity and the derivation of the equations of conics from locus definitions is not required. Rather, the emphasis should be on defining conics in terms of their basic shapes and relating these basic shapes to the parameter values in the standard form and the general form.
- The general form of a quadratic relation is $Ax^2 + Cy^2 + Dx + Ey + F = 0$, or $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where $A, C, D, E, F \in I$ and $B = 0$.
- Students must be able to recognize that degenerate conics are produced when the cutting plane intersects the vertex of a double-napped cone or when the cutting plane intersects a cylindrical region parallel to the generator.
- Degenerate conics will be restricted to visual cases.

Specific Outcomes

Specific Outcome P9-1

Classify conic sections according to shape. [C, R, V]

P9-1 Note:

- Students must consider the conic sections and degenerate conics formed by a plane slicing a double-napped cone or a cylinder.

(See example 1)

Specific Outcome P9-2

Classify conic sections according to a given equation in general or standard (completed square) form (vertical or horizontal axis of symmetry only). [CN, T, V]

P9-2 Notes:

- General form for a quadratic relation is $Ax^2 + Cy^2 + Dx + Ey + F = 0$, or $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where $A, C, D, E, F \in I$ and $B = 0$.
- Standard forms for quadratic relations in completed square form are

$(x - h)^2 + (y - k)^2 = r^2$

$$\left.\begin{array}{l} \dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = +1 \\ \dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = \pm 1 \end{array}\right\} \begin{array}{l} a = \text{horizontal axis} \\ b = \text{vertical axis} \end{array}$$

$y - k = a(x - h)^2$

$x - h = a(y - k)^2$

- Students should be able to determine the type of conic directly from the standard form and from the general form.
- Restrict coefficients in general form to integer values.
- Specific terminology related to conics (other than centre, vertex, domain, range, intercepts, and asymptotes) is not required.
- Students need to understand what an asymptote is but do not need to find the equations of asymptotes of a given hyperbola. When finding the equation of a hyperbola, three of the four parameters a, b, h, and k should be given.
- The location and study of focal points is not required, but students should be able to recognize the centre point of any circle, ellipse, or hyperbola, and the vertex of any parabola.

(See examples 2, 3, and 5)

Specific Outcome P9-3

Convert a given equation of a conic section from general to standard form and vice versa. [R, T]

(See examples 4 and 6)

Acceptable Standard	*Standard of Excellence*
The student can	The student can also
• describe and model the intersection of a double-napped cone and a plane for all conics, including the degenerate cases	• describe and model the intersection of a cylindrical surface and a plane for all conics, including the degenerate cases
• determine the type of conic by examining the equation in general form and in standard form	
• identify domain, range, centre and intercepts of a conic from the equation and from the graph	• determine the type of conic by examining a combination of domain, range, centre, vertices, and intercepts
• sketch the graph of a conic section when given the equation	• sketch the graph of a conic section when given a transformation • sketch the graph of a conic section when given a combination of domain, range, centre, vertices, and intercepts
• describe the effect that changing parameters A and C in the general form of the quadratic relation has on the type of conic produced • describe the effects that changing parameters h, k, a, b in the standard form of the quadratic relation has on the centre and intercepts of the conic	
• write the equation of a conic section when given a graph	• write the equation of a conic section when given a transformation • write the equation of a conic section when given key points such as – vertices – intercepts – centre – point on the curve
• recognize that hyperbolas have asymptotes • convert a given equation of a non-degenerate conic section from general form to standard form and vice versa	
• participate in and contribute toward the problem-solving process for problems that require the analysis of conic sections studied in Pure Mathematics 30	• complete the solution for problems that require the analysis of conic sections studied in Pure Mathematics 30

Examples

Students who achieve the *Acceptable Standard* should be able to answer all the following questions, except for any part of a question labelled SE. Parts labelled SE are appropriate examples for students who achieve the *Standard of Excellence*.

1. A plane intersects a double-napped cone parallel to its axis. If the plane does not pass through the vertex, then the conic produced by this intersection is

A. a circle
B. an ellipse
C. a parabola
D. a hyperbola

Solution

D. a hyperbola

2. Describe the conic defined by $\frac{x^2}{4} + \frac{y^2}{16} = 1$. Provide justification for your answer.

Solution

This is an ellipse because the equation is the sum of squares equal to 1 and $a \neq b$.
The x-intercepts are ± 2, and y-intercepts are ± 4. The ellipse is centred on the origin.
Domain is $-2 \leq x \leq 2$.
Range is $-4 \leq y \leq 4$.

3. Given the equation $x^2 - 4y^2 - 4x = 0$, describe the characteristics of the corresponding graph. Justify your answers.

Solution

Since $AC < 0$, the graph is a hyperbola. Letting $x = 0$ results in the single y-intercept $y = 0$. Letting $y = 0$ results in the two x-intercepts $x = 0$ and $x = 4$. Because of the location of the intercepts, we know the hyperbola opens left and right, has vertices $(4, 0)$ and $(0, 0)$, and is centred at $(2, 0)$. Thus, the domain is $x \leq 0$ or $x \geq 4$. The range is the set of real numbers.

Alternate Solution

The completed square form is $\frac{(x-2)^2}{4} - y^2 = 1$. This tells us the graph is a translation of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and, therefore, is a hyperbola because the equation represents a difference of squares. Since $a^2 = 4$ and $a = 2$, the graph is a translated form of the graph of $x^2 - y^2 = 1$, whose vertices are at $(\pm 1, 0)$. It is stretched horizontally about the y-axis by a factor of two, therefore it is an expansion, so that the new vertices are at $(\pm 2, 0)$. The graph of $\frac{(x-2)^2}{4} - y^2 = 1$ is obtained by translating the graph of $\frac{x^2}{4} - y^2 = 1$ two units to the right so that the vertices are now at $(0, 0)$ and $(4, 0)$. Thus, the domain is $x \leq 0$ or $x \geq 4$. The range is the set of real numbers. The centre is $(2, 0)$.

4. Convert the following equations to standard form and describe the type of conic that each represents.

a. $x^2 + y^2 + 6x - 8y - 11 = 0$

b. $9x^2 + 16y^2 + 64y + 28 = 0$

Solution

a. $(x+3)^2 + (y-4)^2 = 36$

The conic is a circle, with centre at $(-3, 4)$ and radius 6.

b. $\frac{x^2}{4} + \frac{(y+2)^2}{\frac{9}{4}} = 1$

The conic is a horizontal ellipse, with centre at $(0, -2)$.

SE 5. If a hyperbola in standard form of $\frac{(x-h)^2}{9} - \frac{(y-k)^2}{b^2} = -1$ has a centre point of $(2, -1)$ and goes through the point $(2, -2)$, then what is the equation in standard form if $b > 0$?

Solution

$$\frac{(x-h)^2}{9} - \frac{(y-k)^2}{b^2} = -1$$

Substitute point $(2, -2)$ and the centre point $(2, -1)$

$$\frac{(2-2)^2}{9} - \frac{(-2+1)^2}{b^2} = -1$$

$$\frac{(-2+1)^2}{b^2} = -1$$

$$(-1)^2 = b^2$$

$$1 = b^2$$

$$1 = b \text{ since } b > 0$$

$$\frac{(x-2)^2}{9} - \frac{(y+1)^2}{1} = -1$$

SE **6.** **a.** If the ellipse $\frac{(x-4)^2}{9}+\frac{(y+2)^2}{16}=1$ is translated 2 units right and 3 units down, then what is the centre of the translated conic?

SE **b.** Determine the equation of a new ellipse in general form if $x^2+5y^2-25=0$ is translated three units down.

Solution

a. The centre of the ellipse is $(4, -2)$. The translated ellipse will have a centre of $(6, -5)$.

b.

$$x^2+5y^2=25$$

$$\frac{x^2}{25}+\frac{y^2}{5}=1 \qquad \text{(standard form)}$$

$$\frac{x^2}{25}+\frac{(y+3)^2}{5}=1 \qquad \text{(translated down 3 units)}$$

$$x^2+5(y+3)^2=25$$

$$x^2+5(y^2+6y+9)=25$$

$$x^2+5y^2+30y+45=25$$

$$x^2+5y^2+30y+20=0 \qquad \text{(general form)}$$

Permutations and Combinations

General Outcome

Solve problems based on the counting of sets by using techniques such as the Fundamental Counting Principle, permutations, and combinations.

Specific Outcomes

Specific Outcome C6-4

Solve pathway problems, interpreting and applying any constraints. [PS, R]

C6-4 Notes:

- This outcome can be combined with specific outcome P7-14, but only after students have examined permutations and combinations, and Pascal's triangle.
- Illustrative example 4.1 (*Program of Studies*), involves probability theory (addition and multiplication theorems) and is therefore too advanced for the intent of this outcome.
- Students are to find solutions through visualization, the use of the Fundamental Counting Principle, and the use of Pascal's triangle.

(See example 2)

Specific Outcome C6-5

Use the Fundamental Counting Principle to determine the number of different ways of performing multistep operations. [PS, R]

C6-5 Note:

- For this specific outcome, illustrative examples within the *Program of Studies* are appropriate.

(See example 1)

Specific Outcome P7-12

Determine the number of permutations of n different objects taken r at a time, and use the concept of permutations to solve problems. [PS, R, V]

P7-12 Notes:

- Permutation problems should involve repetition and restrictions. (This area may need to be supplemented.)
- Circular and ring permutations are not part of this outcome.

(See examples 3, 4, 5, 6, 11, 18, and 19)

Specific Outcome P7-13

Determine the number of combinations of n different objects taken r at a time, and use the concept of combinations to solve problems. [PS, R, V]

P7-13 Note:

- Students should be familiar with a standard deck of 52 cards (aces are not considered to be face cards).

(See examples 7, 8, 12, 13, 14, 16, and 17)

Specific Outcome P7-14

Determine the number of pathways in a given compound pathway problem. [CN, PS, V]

P7-14 Note:

- More complicated pathways should be solved by using permutations and combinations in some cases (three-dimensional pathways) and using Pascal's triangle in other cases (overlapping pathways).

(See example 15)

Specific Outcome P7-15

Solve problems by using the binomial theorem.

P7-15 Notes:

- In $(a + b)^n$, n belongs to the set of natural numbers.
- Students must explore the relationship between Pascal's triangle and the binomial theorem.
- Students have to have a good working knowledge of all exponent laws.
- There are approved calculators that will do the algebra of many of the examples, such as 9 and 10; therefore, this outcome will be assessed in a different manner on the diploma examination.
- There is a very strong link between this outcome and outcome P7-18 (using the binomial distribution as applied to small samples) in the probability unit.

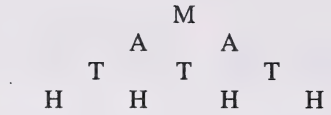
(See examples 9, 10, 20, 21, 22, 23, and 24)

Acceptable Standard	*Standard of Excellence*
The student can	The student can also
• solve simple pathway problems • solve compound two-dimensional pathway problems with a given diagram	• solve three-dimensional pathway problems • solve two-dimensional pathway problems without a given diagram and solve three-dimensional pathway problems with a diagram. They can also generalize the results for an $x \times y$ grid
• apply the Fundamental Counting Principle to various straightforward multistep problems with few constraints	• recognize and address constraints and ambiguity within multistep problems
• understand and use factorial notation • calculate the permutations of n things taken r at a time using ${}_nP_r$	• solve problems involving both permutations and combinations
• obtain partial solutions to problems involving several cases or constraints	• obtain complete solutions to problems involving several cases or constraints
• calculate the number of combinations of n things taken r at a time using ${}_nC_r$	
• solve for n in problems involving ${}_nC_r$ or ${}_nP_r$, given r	• solve for n in problems involving ${}_nC_r$ and ${}_nP_r$, given r
• generate Pascal's triangle and solve related problems	
• expand $(x + y)^n$, where $n \in N$	• expand $(x + y)^n$, where x and y can be monomials involving coefficients and powers
• determine specified terms of an expansion with linear terms within the binomial	• determine specified terms of an expansion with non-linear terms within the binomial • determine the coefficient of a specified term of any binomial expansion
• participate in and contribute toward the problem-solving process for problems involving permutations or combinations studied in Pure Mathematics 30	• complete all solutions toward the problem-solving process for problems involving permutations or combinations studied in Pure Mathematics 30

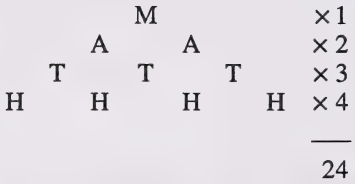
Examples

Students who achieve the *Acceptable Standard* should be able to answer all the following questions, except for any part of a question labelled [SE]. Parts labelled [SE] are appropriate examples for students who achieve the *Standard of Excellence*.

1. Using one letter from each row in the following diagram, determine the number of outcomes where the four letter words spell **MATH**.



Solution

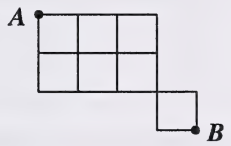


Using the Fundamental Counting Principle, there are 24 possible outcomes that spell **MATH**.

2. (From Pure Mathematics 30 January 2001 Diploma Examination, Multiple-Choice Question 26)

Use the following information to answer the next question.

A student must draw a path from point A to point B in the diagram below.



If each path must be drawn along the lines such that it is always getting closer to B, then the number of paths that the student can draw is

A. 12
* **B.** 20
C. 40
D. 70

Solution

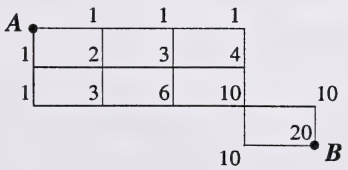
To get from A to the intersection point, the path must be drawn only 5 block lengths.

Any path from A to intersection is comprised of 3 blocks to the right (3Rs) and 2 blocks down (2Ds) or $\frac{5!}{3!\,2!} = 10$.

To get from the intersection point to B, the path must be drawn along 2 block lengths (1R) and (1D) or $\frac{2!}{1!\,1!} = 2$. To get from A to B $= 10 \times 2 = 20$.

Alternate Solution

Treat the grid as a portion of Pascal's triangle, with its apex at A. Then, put a number at each corner to represent the number of ways to get to that corner. There are 20 paths from A to B.



3. An airline pilot reported that in seven consecutive days she spent, in an unspecified order, one day in Winnipeg, one day in Regina, two days in Edmonton, and three days in Yellowknife. How many different itineraries were possible? How many itineraries were possible if she spent the first day and the last day in Yellowknife?

Solution

Treat cities as letters and use 7 letters for 7 days: WREEYYY
Possible itineraries equal possible arrangements.

Number of itineraries = $\frac{7!}{2!3!} = \frac{5\,040}{12} = 420$ itineraries

If the first day and the last day had to be spent in Yellowknife, the number of itineraries would be $\frac{5!}{2!} = \frac{120}{2} = 60$ itineraries.

4. (From Pure Mathematics 30 January 2001 Diploma Examination, Multiple-Choice Question 24)

If all of the letters in the word **DIPLOMA** are used, then the number of different 7-letter arrangements that can be made beginning with 3 vowels is

A. 24
* **B.** 144
C. 720
D. 5 040

Solution

$3!4! = 144$

5. An executive consisting of a president, vice-president, treasurer, and secretary must be formed from a group of 20 people. Calculate the number of executives possible.

Solution

P, VP, T, S
The positions are distinguishable, so ${}_{20}P_4 = 116\,280$.

6. Explain the meaning of ${}_8P_3$. Why does ${}_3P_8$ not make sense?

Solution

${}_8P_3$ is the number of arrangements of 8 objects, taken 3 at a time.

${}_3P_8$ would be the number of arrangements of 3 objects, taken 8 at a time, which is impossible.

7. From a group of 5 student representatives, 3 will be chosen to work on a dance committee.

a. List all possible committees.

b. Calculate ${}_5C_3$, and compare the solution with the answer in part a.

SE **c.** If the committee had to have a chairperson, would this still be a combination problem? Why or why not?

SE **d.** How many committees of 3, with a chairperson, can be chosen from a group of 10 student representatives?

Solution

a. Students are *A*, *B*, *C*, *D*, and *E*. Committees of 3 are in any order.

ABC *ACD* *ADE* *BCD* *BDE* *CDE*
ABD *ACE* *BCE*
ABE

b. ${}_5C_3 = \dfrac{5!}{(5-3)!3!} = \dfrac{120}{2 \times 6} = 10$

SE **c.** Yes, because once a chairperson is selected, the remainder of the committee would be chosen by combinations. No other position is specified.

SE **d.** $10 \times {}_9C_2 = 10 \times 36 = 360$

8. How many diagonals are there in a regular polygon with 20 sides? What is the general formula for the number of diagonals in an *n*-sided polygon?

Solution

${}_{20}C_2 - 20$ sides $= 190 - 20 = 170$

${}_nC_2 - n$ diagonals in an *n*-sided polygon

9. In the expansion of $(2x-3)^6$, what is the term that contains x^3 ?

Solution

$${}_6C_3(2x)^3(-3)^3 = 20(8x^3)(-27) = -4\,320x^3$$

10. Expand $(x+7)^7$ by using the binomial theorem.

Solution

$$\begin{aligned}&{}_7C_0(x^7) + {}_7C_1(x^6)(7) + {}_7C_2(x^5)(7^2) + {}_7C_3(x^4)(7^3) + {}_7C_4(x^3)(7^4)\\&+ {}_7C_5(x^2)(7^5) + {}_7C_6(x)(7^6) + {}_7C_7(7^7)\\= {}& x^7 + 49x^6 + 1029x^5 + 12\,005x^4 + 84\,035x^3 + 352\,947x^2 + 823\,543x + 823\,543\end{aligned}$$

11. Algebraically solve ${}_nP_2 = 30$ for n.

Solution

$$\frac{n!}{(n-2)!} = 30$$

$$\frac{n(n-1)(n-2)!}{(n-2)!} = 30$$

$$\begin{aligned}n(n-1) &= 30\\n^2 - n - 30 &= 0\\(n-6)(n+5) &= 0\\n &= 6\end{aligned}$$

12. If ${}_nC_2 = 15$, what is n?

Solution

$${}_nC_2 = 15$$

$$\begin{aligned}\frac{n!}{(n-2)!2!} &= 15\\\frac{n(n-1)}{2} &= 15\\n^2 - n &= 30\\n^2 - n - 30 &= 0\\(n-6)(n+5) &= 0\\n &= 6\end{aligned}$$

SE **13.** Solve ${}_nC_2 = \frac{{}_nP_3}{3!}$, and verify your solution.

Solution

$$\frac{n(n-1)(n-2)!}{(n-2)!\,2!} = \frac{n(n-1)(n-2)(n-3)!}{(n-3)!\,3!}$$

$$\begin{aligned} n(n-1)3! &= n(n-1)(n-2)2! \\ 3! &= (n-2)2! \\ 6 &= 2n-4 \\ 10 &= 2n \\ n &= 5 \end{aligned}$$

LHS	RHS
${}_5C_2 = 10$	$\frac{{}_5P_3}{3!} = \frac{60}{6} = 10$

Students who achieve the *Standard of Excellence* should also be able to answer the following questions.

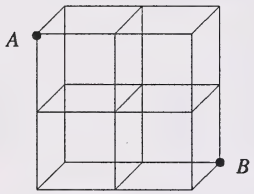
SE **14.** In a group of people, there are 10 females and 12 males. Determine the number of 4-member committees consisting of at least 1 female that can be formed.

Solution

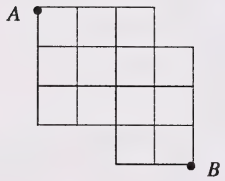
Total number of committees possible – number committees with no females.

$_{22}C_4 - {}_{12}C_4 = 7\,315 - 495 = 6\,820$

SE **15.** **a.** Determine the number of paths possible from point A to point B in the following diagram if travel may occur only along the edges of the cubes and if the path must always move closer to B.

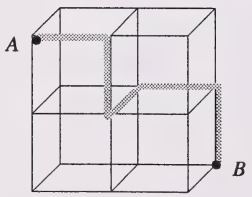


b. Find the number of paths from point A to point B, if paths must always move closer to B.



Solution

a.



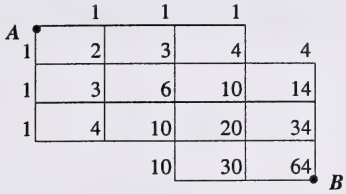
One possible path is shown. Any path from A to B must travel along 5 edges and must consist of 2 edges to the right (R), 2 edges downward (D), and 1 edge to the back (B).

The number of arrangements of these letters is

$$\frac{5!}{2!2!} = \frac{120}{4} = 30.$$

Therefore, 30 paths are possible.

b. Count number of paths to each corner in grid, as in Pascal's triangle.



$\therefore$ There are 64 pathways from A to B.

SE **16.** Chris is one of a group of 10 people from which a committee of 3 people must be formed. Determine the number of different committees possible if Chris must sit on the committee.

Solution

$_9C_2 = 36$

SE **17.** Show that $_nC_k = {}_n C_{n-k}$. Verify this equation for the case where $n = 10$ and $k = 3$.

Solution

$$_nC_k = \frac{n!}{(n-k)!k!} \qquad _nC_{(n-k)} = \frac{n!}{(n-(n-k))!(n-k)!}$$

$$= \frac{n!}{k!(n-k)!}$$

$$_{10}C_3 = \frac{10!}{3!7!} = 120 \qquad _{10}C_7 = \frac{10!}{7!3!} = 120$$

SE **18.** Develop and solve a problem where $_8P_3$ would be applicable.

Solution

Answers will vary.

SE **19.** A teacher tells his students that on a multiple-choice test of 12 questions, 3 answers are **A**, 3 are **B**, 3 are **C**, and 3 are **D**. How many different answer keys are possible?

Solution

$$\frac{12!}{3!3!3!3!} = 369\,600$$

SE **20.** Expand $\left(2a - \frac{3}{b}\right)^4$ by using the binomial theorem.

Solution

$$16a^4 - \frac{96a^3}{b} + \frac{216a^2}{b^2} - \frac{216a}{b^3} + \frac{81}{b^4}$$

SE **21.** In the expansion $(2x^2 - 3)^{15}$, what is the coefficient of the simplified term involving x^8 ?

Solution

$$\begin{aligned} {}_{15}C_{11}(2x^2)^4(-3)^{11} &= 1365(16x^8)(-177\,147) \\ &= -3\,868\,890\,480x^8 \end{aligned}$$

$\therefore$ coefficient is $-3\,868\,890\,480$

SE **22.** Determine the constant term in the expansion $\left(2x^2 - \frac{1}{x}\right)^6$.

Solution

$$\begin{aligned} t_{k+1} &= {}_nC_k p^{n-k} q^k \\ &= {}_6C_k (2x^2)^{6-k} \left(-\frac{1}{x}\right)^k \\ &= {}_6C_k 2^{6-k} x^{12-2k} \frac{(-1)^k}{x^k} \\ &= {}_6C_k 2^{6-k} (-1)^k x^{12-3k} \end{aligned}$$

x^0 implies a constant

$$\begin{aligned} \therefore\ 12 - 3k &= 0, \\ 12 &= 3k \\ 4 &= k \end{aligned}$$

$$\begin{aligned} \therefore\ {}_6C_4 (2x^2)^2 \left(-\frac{1}{x}\right)^4 &= 15(4x^4)\left(\frac{1}{x^4}\right) \\ &= 60 \end{aligned}$$

23. Investigate the sample space for flipping 1 coin, 2 coins, 3 coins, 4 coins, ... n coins, and make an organized list of outcomes. Relate this organized list to Pascal's triangle and the binomial theorem.

Solution

# of Coins	**1**	**2**	**3**	**4**	...	***n***
Outcomes	H, T	HT, TH, HH, TT	HHH, HHT, HTH, THH, TTT, TTH, THT, HTT			
# of Outcomes	2	4	8	16	...	2^n

The number of outcomes represents the sum of the $(n + 1)^{st}$ row of Pascal's triangle, which can be expressed using the binomial theorem.

24. (From Pure Mathematics 30 January 2001 Diploma Examination, Numerical-Response Question 4)

If one term in the expansion of $(x-b)^{10}$, $b>0$, is $\frac{76\,545}{32}x^4$, then the value of b, correct to the nearest tenth, is _____.

Solution

$$
\begin{aligned}
{}_{10}C_6(x)^{10-6}(-b)^6 &= \frac{76\,545}{32}x^4 \\
210x^4b^6 &= \frac{76\,545}{32}x^4 \\
b^6 &= \frac{76\,545}{32(210)} \\
b^6 &= \frac{729}{64} \\
b &= \sqrt[6]{\frac{729}{64}} \\
&= \frac{3}{2} \\
&= 1.5
\end{aligned}
$$

Probability

General Outcome

Model the probability of a compound event and solve problems based on the combining of simpler probabilities.

General Notes:

- The notation $P(\overline{A})$ means the probability of "not A," the complement of A.
- Particular attention must be paid to the usage of the words "and" and "or" in probability questions.
- Teachers should consider using supplementary materials.
- Routine problems include independent events, dependent events, mutually exclusive events, and exclusive events.
- Teachers should be aware of the *Statistics* unit outcomes when teaching the *Probability* unit because the outcomes are related.

Specific Outcomes

Specific Outcome C6-6

Construct a sample space for two or three events. [PS, R, V]

C6-6 Notes:

- The intent of this outcome is that students can construct a sample for an experiment that could contain up to three components; for example, tossing a die and flipping a coin, or flipping three coins.
- Students should be able to determine probability based on sample spaces using tree diagrams, Venn Diagrams, or the Fundamental Counting Principle.

(See examples 1 and 2)

Specific Outcome C6-7

Classify events as either independent or dependent events. [C]

C6-7 Notes:

- Dependent events are used in determining the conditional probabilities required in P7-16.
- Students should be able to solve routine problems involving independent or dependent events.

  Dependent Events $P(A \text{ and } B) = P(A) \times P(B|A)$

  Independent Events $P(A \text{ and } B) = P(A) \times P(B)$

(See examples 3, 4, and 5)

Specific Outcome C6-8

Solve problems by using the probabilities of mutually exclusive and complementary events. [CN, PS, R]

C6-8 Note:

- Within this outcome, it is expected that students will also solve problems involving mutually and non-mutually exclusive events.

  Mutually Exclusive Events $P(A \text{ or } B) = P(A) + P(B)$

  Non-mutually Exclusive Events $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

(See examples 6, 7, 8, 9, 10, and 11)

Specific Outcome P7-16

Determine the conditional probability of two events. (Bayes' Law) [E, PS, R]

P7-16 Notes:

- These probabilities can also be calculated using permutations and combinations, which are covered by P7-17.

(See examples 12 and 13)

Specific Outcome P7-17

Solve probability problems involving permutations, combinations, and conditional probability. [E, PS, R]

P7-17 Note:

- Students will be solving the same types of problems as in P7-16 but will use permutations and combinations.

(See examples 14, 15, 16, and 17)

Specific Outcome P7-18 [PS, R, T]

Solve probability problems by using the binomial distribution as applied to small samples.

P7-18 Notes:

- A binomial distribution is considered a small sample if $np < 5$ **or** $n(1-p) < 5$, where n is the number of trials and p is the probability of success.
- The Binomial Theorem should be used if the problem involves a small sample.
- A binomial experiment is one that consists of n identical trials where each trial results in one of two outcomes. The probability of success of any trial remains constant throughout the experiment. The sum of the probabilities of these outcomes is 1 and the trials are independent.
- The Binomial Theorem should also be used if the problem involves a short interval.
- A binomial distribution is the sequence of all possible probabilities resulting from a binomial experiment.
- When both $np \geq 5$ **and** $n(1-p) \geq 5$, the sample is considered large enough and probabilities should be approximated by using the normal distribution.

(See examples 18, 19, 20, and 22)

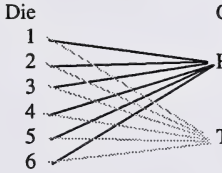
Acceptable Standard	*Standard of Excellence*
The student can	The student can also
• determine sample spaces using common probability situations such as rolling dice, tossing coins, and drawing cards for experiments involving only two outcomes • determine sample spaces for non-routine problems of two or three components • distinguish between independent and dependent events • distinguish between mutually exclusive and non-mutually exclusive events • solve problems involving mutually exclusive events • solve problems involving non-mutually exclusive events • determine the complement to a particular event and solve problems involving complemetary events • determine the probability of one event in which permutations and combinations are involved	
• determine probabilities for two events for routine problems, in which permutations and combinations are not involved	• determine probabilities for two events in which permutations and combinations are involved • determine answers for non-routine problems having two events
• determine probabilities of routine problems	• determine probabilities of non-routine problems, including questions requiring knowledge of infinite geometric sequences
• determine probability using the binomial distribution for small samples for a single trial and for exact number of successes on multiple trials	• determine probabilities using the binomial distribution for a multiple-trial event and for an inexact (at least, at most) number of successes over multiple trials
• participate in and contribute toward the problem-solving process for probability problems studied in Pure Mathematics 30	• complete the solution to probability problems studied in Pure Mathematics 30

Examples

Students who achieve the *Acceptable Standard* should be able to answer all the following questions, except for any part of a question labelled SE. Parts labelled SE are appropriate examples for students who achieve the *Standard of Excellence*.

1. List the sample space for rolling a 6-sided die and flipping a coin.

Solution



1 and H	3 and H	5 and H
1 and T	3 and T	5 and T
2 and H	4 and H	6 and H
2 and T	4 and T	6 and T

2. Draw or list the sample space for the following situation. A bus is scheduled to arrive at a train station at any time between 07:05 and 07:15 inclusive. A train is scheduled to arrive between 07:11 and 07:17 inclusive. The arrival of a bus at 07:06 and a train at 07:14 can be represented by the point (6, 14). Times are expressed as whole minutes.

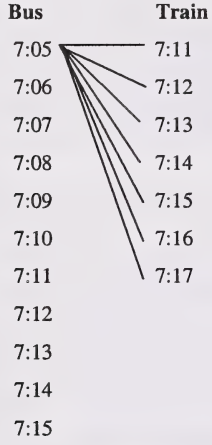
a. How many points are there in this sample space?

b. How many points have the bus and the train arriving at the same time?

c. How many points have the bus arriving after the train?

d. What is the probability of the bus arriving after the train?

Solution



a. $11 \times 7 = 77$

b. 5 points

c. $1 + 2 + 3 + 4 = 10$ points

d. $P(\text{bus after train}) = \frac{10}{77}$

		Train						
	Time	:11	:12	:13:	:14	:15	:16	:17
Bus	:05							
	:06							
	:07							
	:08							
	:09							
	:10							
	:11							
	:12	X						
	:13	X	X					
	:14	X	X	X				
	:15	X	X	X	X			

3. Classify the following events as independent or dependent.

a. Tossing "heads" in a coin toss *and* rolling a 6 on a die

b. Out of a deck of 52 cards, the first card drawn is an ace, and the second card drawn is an ace

c. Out of a deck of 52 cards, the first card drawn is a king, and after replacement, the second card drawn is a queen

Solution

a. Independent

b. Dependent

c. Independent

4. A bag contains 6 red marbles and 10 green marbles. From the bag, 2 marbles are drawn without replacement. Determine the probability of drawing

a. 2 red marbles

b. 1 red and then 1 green marble

c. 1 red and 1 green marble

Solution

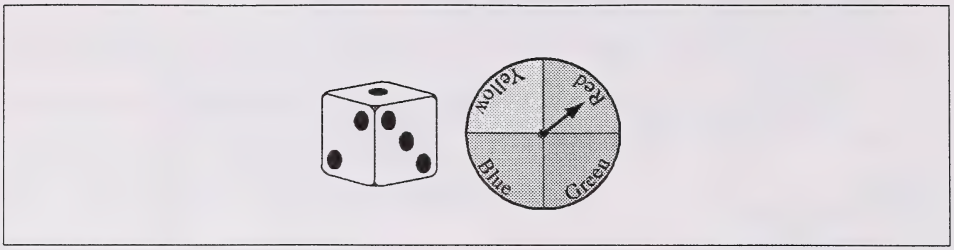
a. $P(\text{2 red}) = \frac{6}{16} \times \frac{5}{15} = \frac{1}{8}$

b. $P(\text{1 red then 1 green}) = \frac{6}{16} \times \frac{10}{15} = \frac{1}{4}$

c. $P(\text{1 red and 1 green}) = \frac{6}{16} \times \frac{10}{15} + \frac{10}{16} \times \frac{6}{15}$

$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

Use the following information to answer the next question.



5. A person has a resular six-sided die and a spinner that has been divided into 4 equal parts. Determine the probability of

a. the spinner anding in the area marked blue and rolling a 3 on the die

b. the spinner anding in the area marked yellow and rolling an even number on the die

c. the spinner **not** landing in the area marked blue and rolling a prime number on the die

Solution

a. $P(\text{Blue and 3}) = \frac{1}{4} \times \frac{1}{6} = \frac{1}{24}$

b. $P(\text{Yellow and Even Number}) = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$

c. $P(\text{Not Blue and Prime Number}) = \frac{3}{4} \times \frac{2}{3} = \frac{1}{2}$

6. A single card is drawn from a standard deck of 52 playing cards.
Event A is drawing a spade.
Event B is drawing a red card.
Classify events A and B as either mutually exclusive or as not mutually exclusive events.

Solution

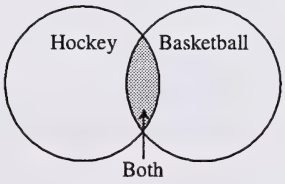
Mutually exclusive, since spades and red cards have no cards in common
(i.e., spades are black cards).

7. In a senior class, 32% of the students play hockey and 41% of the students play basketball. Of the students who play hockey and the students who play basketball, 10% play both hockey and basketball.

What is the probability that a student in the senior class plays both basketball or hockey?

Solution

Venn Diagram



or

$$\begin{aligned} P(B \text{ or } H) &= P(B) + P(H) - P(B \text{ and } H) \\ &= 0.41 + 0.32 - 0.10 \\ &= 0.63 \end{aligned}$$

8. A single card is drawn from a standard deck of 52 playing cards.
Event A is drawing a heart.
Event B is drawing a red card.
Classify events A and B as either mutually exclusive or non-mutually exclusive events.

Solution

Non-mutually exclusive, since hearts and red cards have cards in common
(i.e., all hearts are red cards).

9. A (single) die is rolled. What is the probability that a 2 or a 6 is rolled?

Solution

$$\begin{aligned} P(2 \text{ or } 6) &= P(2) + P(6) - P(2 \text{ and } 6) \\ &= \frac{1}{6} + \frac{1}{6} - 0 \\ &= \frac{1}{3} \end{aligned}$$

10. If the probability of winning a game is $\frac{1}{31}$ and a tie is not possible, what is the probability of losing the game?

Solution

$$P(\text{win}) = \frac{1}{31}$$

$$P(\text{lose}) = 1 - \frac{1}{31} = \frac{30}{31}$$

11. A single card is drawn from a standard deck of cards. What is the probability of drawing a red card or a jack?

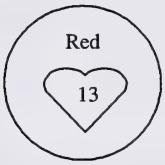
Solution

$$\begin{aligned} P(\text{red or jack}) &= P(\text{red}) + P(\text{jack}) - P(\text{red and jack}) \\ &= \frac{26}{52} + \frac{4}{52} - \frac{2}{52} \\ &= \frac{28}{52} \\ &= \frac{7}{13} \end{aligned}$$

12. A single card is drawn from a standard deck of 52 cards. What is the probability that the card is a heart, given that the card is red?

Solution

Venn Diagram



or

Bayes' Law

$$P(\text{heart} \mid \text{red}) = \frac{13}{26} = \frac{1}{2}$$

$$= \frac{P(\text{red and heart})}{P(\text{red})}$$

$$= \frac{\frac{13}{52}}{\frac{26}{32}}$$

$$= \frac{13}{26} = \frac{1}{2}$$

or

Chart

	Red	Black	Total
Heart	13*	0	13
Not Heart	13	26	39
Total	26^	26	26

$$P(\text{heart} \mid \text{red}) = \frac{P(\text{red and heart})^*}{P(\text{red})^\wedge}$$

$$= \frac{13}{26}$$

$$= \frac{1}{2}$$

13. It is known that 10% of a population has a certain disease. For a person without the disease, a blood test for the disease shows "not positive" 95% of the time. For a person with the disease, the blood test shows "positive" 99% of the time. What is the probability that a person whose blood test shows positive for the disease actually has the disease?

Solution

P(has disease | shows positive as having disease)

$$= \frac{10\% \times 99\%}{\underbrace{10\% \times 99\%}_{\text{diseased and showing positive}} + \underbrace{90\% \times 5\%}_{\text{not diseased but showing positive}}} = \frac{0.1 \times 0.99}{0.1 \times 0.99 + 0.9 \times 0.05} = 0.6875$$

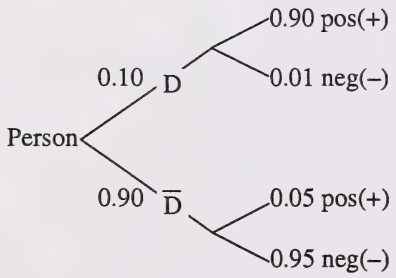
Alternate Solution

Assume that there are 1 000 people.

		Blood Test		
		Not positive	Positive	Total
Actual Condition	Not Diseased	855	45	900
	Diseased	1	99	100
	Total	856	144	1 000

Probability of diseased and showing positive $= \frac{99}{144} = 0.687$

Alternate Solution



$$P(\text{D}|+) = \frac{P(\text{D and } +)}{P(+)}$$

$$= \frac{0.99}{(0.01 \times 0.99) + (0.90 \times 0.05)}$$

$$= 0.687$$

14. Five books, each of a different colour, and including one red and one green book, are randomly placed on a shelf. What is the probability of the red book being at one end and the green book at the other end?

Solution

$$P(\text{R}___\text{G}) \text{ or } P(\text{G}___\text{R}) = \left(\frac{1 \times 3! \times 1}{5!}\right) \times 2 = \frac{12}{120} = \frac{1}{10}$$

15. What is the probability of being dealt all 4 aces in a 5-card hand dealt from a standard 52-card deck?

Solution

$$P(4 \text{ Aces}) = \frac{{}_4C_4 \times {}_{48}C_1}{{}_{52}C_5} = \frac{48}{2\,598\,960} = 0.0000185$$

16. A bag contains 6 red marbles and 10 green marbles. From the bag, 2 marbles are drawn without replacement. Determine the probability of drawing

a. 2 red marbles

b. 1 red and then 1 green marble

c. 1 red and 1 green marble

d. at least 1 green marble

Solution

a. 2 red marbles

$$P(\text{2 red}) = \frac{{}_6C_2 \times {}_{10}C_0}{{}_{16}C_2} = \frac{1}{8}$$

b. 1 red and then 1 green marble

$$P(\text{1 red then 1 green}) = \frac{{}_6P_1 \times {}_{10}P_1}{{}_{16}P_2} = \frac{1}{4}$$

c. 1 red and 1 green marble

$$P(\text{1 red and 1 green}) = \frac{{}_6C_1 \times {}_{10}C_1}{{}_{16}C_2} = \frac{1}{2}$$

d. at least 1 green marble

$$P(\text{at least 1 green}) = \frac{{}_6C_1 \times {}_{10}C_1 + {}_{10}C_2 \times {}_6C_0 +}{{}_{10}C_2} = \frac{105}{120}$$

or

$$\begin{aligned} &= 1 - P(\text{no green}) \\ &= 1 - \frac{{}_6C_2}{{}_{16}C_2} \\ &= 1 - \frac{15}{120} \\ &= \frac{105}{120} \end{aligned}$$

17. In hockey, a "shootout" consists of teams A and B taking alternate shots on goal. Each team shoots once in each round. If both teams score, or both teams miss, they must go to another round. On a particular shootout, team A has a probability of 0.3 of scoring with any one shot and team B has a probability of 0.4 of scoring with any one shot.

a. If Team A shoots first, what is the probability of Team B winning on its first shot?

b. If Team A shoots first, what is the probability of Team A winning on its first shot?

SE **c.** What is the probability of Team A winning on the second round?

SE **d.** What is the probability of Team B winning on the second round?

SE **e.** What is the probability of Team A winning?

Solution

a. $P(\overline{A}) = 1 - 0.3 = 0.7,\ P(B) = 0.4$

$$P(\overline{A} \text{ and } B) = 0.7 \times 0.4$$
$$= 0.28$$

b. $$P(A) = 0.3 \times P(\overline{B}) = 0.3 \times 0.6 = 0.18$$

SE **c.** $P(A$ wins in 2 rounds of shots)

$$= \left[\underbrace{0.3 \times 0.4}_{\text{Both Score}} + \underbrace{0.7 \times 0.6}_{\text{Both Miss}}\right] \times \left[\underbrace{0.3 \times 0.6}_{A \text{ Scores } B \text{ Misses}}\right]$$
$$= [0.12 + 0.42] \times 0.18$$
$$= 0.54 \times 0.18$$
$$= 0.0972$$

SE **d.** $P(B \text{ wins in } 2 \text{ rounds of shots})$

$$= \underbrace{\left[\underbrace{0.3 \times 0.4}_{\text{Both Score}} + \underbrace{0.7 \times 0.6}_{\text{Both Miss}}\right]}_{\textbf{Round 1}} \times \underbrace{\left[\underbrace{0.7 \times 0.4}_{A \text{ Misses } B \text{ Scores}}\right]}_{\textbf{Round 2}}$$

$$= [0.12 + 0.42] \times 0.28$$
$$= 0.54 \times 0.28$$
$$= 0.1512$$

SE **e.** $P(\text{Team } A \text{ wins})$

$$= \underbrace{\left[\begin{array}{c}\text{Team } A \\ \text{wins}\end{array}\right]}_{\textbf{Round 1}} + \underbrace{\left[\begin{array}{c}\text{Neither team wins in} \\ \text{Round 1 and} \\ \text{Team } A \text{ wins Round 2}\end{array}\right]}_{\textbf{Round 2}} + \underbrace{\left[\begin{array}{c}\text{Neither team wins in} \\ \text{Rounds 1 and 2, and} \\ \text{Team } A \text{ wins Round 3}\end{array}\right]}_{\textbf{Round 3}} + \ldots$$

$$= [0.3 \times 0.6] + [(0.3 \times 0.4) + (0.7 \times 0.6) + (0.3 \times 0.6)]$$
$$+ [(0.3 \times 0.4) + (0.7 \times 0.6)^2 \times 0.3 \times 0.6] + \ldots$$
$$= \quad \bullet \quad \bullet \quad \bullet$$
$$= \quad \bullet \quad \bullet \quad \bullet$$
$$= \quad \bullet \quad \bullet \quad \bullet$$

or

$$P(A \text{ wins}) = \frac{a}{1 - r}$$
$$= \frac{0.3 \times 0.6}{1 - (0.3 \times 0.4 + 0.7 \times 0.6)}$$
$$= 0.3913$$

18. A quiz consists of 6 multiple-choice questions with 4 choices for each question.

a. Determine the probability that a student will guess correctly for any single question.

b. Determine the probability that a student will guess incorrectly for any single question.

c. Determine the sum of these two probabilities.

d. Determine the probability of getting exactly 50% on this test by guessing alone.

SE **e.** Determine the probability of passing the test (3 or more questions correct) by guessing alone.

Solution

a. $P(C) = \frac{1}{4}$

b. $P(\overline{C}) = 1 - \frac{1}{4} = \frac{3}{4}$

c. $\frac{1}{4} + \frac{3}{4} = 1$

d. $P(50\%)$

$$\left.\begin{aligned} np &= 6 \times 0.25 = 1.5 < 5 \\ n(1-p) &= 6 \times 0.75 = 4.5 < 5 \end{aligned}\right\} \therefore \text{ small sample, use binomial distribution}$$

$$P(50\%) = {}_6C_3(0.25)^3(0.75)^3 \doteq 0.13$$

SE **e.** $P(\text{Pass})$

$$= {}_6C_3(0.25)^3(0.75)^3 + {}_6C_4(0.25)^4(0.75)^2 + {}_6C_5(0.25)^5(0.75)^1 + {}_6C_6(0.25)^6$$
$$\doteq 0.169$$

19. A baseball player has a batting average of 0.300.

a. Determine the probability of the player getting exactly 1 hit at her next 3 times at bat.

SE **b.** Determine the probability of the player getting at least 1 hit at her next 3 times at bat.

Solution

a. $\left.\begin{array}{l} np = 3 \times 0.3 = 0.9 < 5 \\ n(1-p) = 3 \times 0.7 = 2.1 < 5 \end{array}\right\} \therefore$ small sample, use binomial distribution

$$P(1) = {}_3C_1(0.3)^1(0.7)^2$$
$$= 0.441$$

SE **b.** $P(\text{at least } 1) = {}_3C_1(0.3)^1(0.7)^2 + {}_3C_2(0.3)^2(0.7)^1 + {}_3C_3(0.3)^3$
$$= 0.657$$

20. A family has nine children. If, for each birth, there is an equal likelihood that a boy or a girl is born, what is the probability that there are 7 boys and 2 girls?

Solution

$$P(7B \text{ and } 2G) = {}_9C_7(0.5)^7(0.5)^2$$
$$\textbf{or} = {}_9C_2(0.5)^2(0.5)^7 \doteq 0.07$$

21. In a particular country, the probability of a child being a girl is 0.510. A family of 5 children is known to have at least 2 girls. What is the probability of this family having exactly 4 girls?

Solution

$P(4G)$

$$= \frac{{}_5C_4(0.510)^4(0.490)^1}{{}_5C_2(0.510)^2(0.490)^3 + {}_5C_3(0.510)^3(0.490)^2 + {}_5C_4(0.510)^4(0.490)^1 + {}_5C_5(0.510)^5}$$

$$\doteq 0.201$$

SE **22.** A particular exam consists of 25 multiple-choice questons. For each question there are 4 possible choices. What is the probability that a student will get at least 22 of the questions correct by guessing?

Solution

$P(\text{at least } 22)$

$$= {}_{25}C_{22}\left(\frac{1}{4}\right)^{22}\left(\frac{3}{4}\right)^{3} + {}_{25}C_{23}\left(\frac{1}{4}\right)^{23}\left(\frac{3}{4}\right)^{2} + {}_{25}C_{24}\left(\frac{1}{4}\right)^{24}\left(\frac{3}{4}\right)^{1} + {}_{25}C_{25}\left(\frac{1}{4}\right)^{25}\left(\frac{3}{4}\right)^{0}$$

$$= 5.762 \times 10^{-11}$$

Note:

$$np > 5 \qquad n(1-p)5 > 5$$
$$25(0.25) > 5 \qquad 25(1-0.25) > 5$$

In this example, even though $np > 5$ and $n(1-p) > 5$, the length of each interval is less than 5 (22 to 25), therefore binomial distribution should be used to find the best estimate.

Statistics

General Outcome

Use normal and binomial probability distributions to solve problems involving uncertainty.

General Notes:

- The students should be using their calculators to enter data and calculate statistical solutions.
- Review measures of central tendency and histograms. Teachers may also need to review the difference between a sample and a population.
- A probability distribution can be displayed as a histogram that gives the probability for every outcome of an experiment.
- A probability distribution can be converted into a data set. For example, a deck of cards contains: 2, 3, 4, 5, 6, 7, 8, 9, 10, jack, queen, king, and ace, in four different suits. The jack, queen, king, and ace each have a value of 10 and each of thse cards is worth its respective face value. Find the mean and standard deviation of this set of values. (Enter the numbers 2, 3, 4, 5, 6, 7, 8, 9, 10, 10, 10, 10, 10 into a list. The mean is 7.23, and the standard deviation is 2.83.)
- Binomial probability distributions **should** be approximated by a normal distribution whenever $np \geq 5$, $n(1-p) \geq 5$, and the interval length is larger than 5.
- The continuity correction is not part of the curriculum.
- All data are considered to be discrete.
- If asked to find an **exact** number of successes in a probability experiment, the binomial distribution method should be used (see P7-18).
- Review *Probability* outcomes because they are related.
- Data distribution may include histograms, tabled data, frequency polygons, and raw data.

Specific Outcomes

Specific Outcome C6-1

Find the population standard deviation of a data set or a probability distribution by using technology. [CN, E, T, V]

C6-1 Note:

- Sample standard deviation will not be used. Population mean is μ and standard deviation is represented by σ. Some resources use $\bar{x}$ for the population mean.

(See example 2)

Specific Outcome C6-2

Use z-scores and z-score tables to solve problems. [PS, R, T, V]

C6-2 Notes:

- The z-score tables are organized differently in the two recommended resources.
- Teachers need to be aware that the z-scores from the textbook and data table vary from the z-scores on a calculator.
- Some graphing calculators on the approved list do not have the capability of calculating and drawing z-score probability curves; therefore, tables are necessary with these models of calculators.

(See examples 1, 3, and 5)

Specific Outcome C6-3

Use the normal distribution and the normal approximation to the binomial distribution to solve problems involving confidence intervals for large samples. [CN, E, PS]

C6-3 Notes:

- Students should be able to determine the mean $(\mu = np)$ and standard deviation $\left(\sigma = \sqrt{np(1-p)}\right)$ of a binomial distribution.
- Margin of error is not considered to be part of the curriculum.
- When choosing intervals, make sure to select intervals that are large (length greater than 5) so that continuity corrections can be ignored. Also, make sure large samples are used for the same reason. (See notes in Probability, P7-18)
- Continuity correction is not considered to be part of the curriculum.
- When calculating bounds for confidence intervals, to ensure that the minimum percentage of data is correctly included, students should round down at the lower bound and round up at the upper bound.

(See examples 4 and 6)

Acceptable Standard	*Standard of Excellence*
The student can	The student can also
• determine standard deviation and mean using technology	
• answer questions related to given data and compare with normal distribution	
• rank standard deviations without any calculations, given various data distributions	• interpret how changing data can affect the standard deviation and/or mean
• find areas under the curve when given μ and σ	
• calculate z-scores by using the formula	
• calculate the missing value when given the z-score and either μ or σ	
• calculate the area between any two z-scores	
• use z-scores to compare two sets of data and draw conclusions	
• given a binomial distribution, calculate mean and standard deviation by using the appropriate formulas	• calculate a z-score and apply it to the normal curve, given binomial distribution data
• calculate the endpoint values for a given confidence interval	• given a specified area under the curve and one z-score, find the other z-score
• discuss the effects of increasing and decreasing sample size on standard deviation and mean (Students may use formulas here.)	
• participate in and contribute toward the problem-solving process for problems that require the analysis of statistics studied in Pure Mathematics 30	• complete the solution to problems that require the analysis of statistics studied in Pure Mathematics 30

Examples

Students who achieve the *Acceptable Standard* should be able to answer all the following questions, except for any part of a question labelled SE. Parts labelled SE are appropriate examples for students who achieve the *Standard of Excellence*.

1. The body temperature of a sample of 122 people was taken. The 122 people were found to have a mean body temperature of 36.8°C, with a standard deviation of 0.35°C.
If a normal distribution is assumed, what is the expected number of people with a body temperature

a. above 37.0°C?

b. below 36.0°C?

Solution

a. $z_{37} = \dfrac{37 - 36.8}{0.35} = 0.57$, $1 - 0.7157 = 0.2843$ $0.2843 \times 122 \doteq 35$ people

b. $z_{36} = \dfrac{36 - 36.8}{0.35} = -2.29$, $0.11 \times 122 \doteq 1$ person

2. A set of 100 numbers has a standard deviation of 10. To each of the numbers, 5 is added.

a. What effect does this addition have on the standard deviation?

SE **b.** How could the standard deviation be increased to 12?

Solution

a. No effect on standard deviation.

SE **b.** Change all data values to z-scores. Then, using these z-scores, the same mean, and a standard deviation of 12, calculate the new values.

3. Jack has a test score of 75% on a test with a mean of 65%. To get into college, he needs to be in the top 10% of those who took the test. What is the largest standard deviation that will allow Jack to be in the top 10%?

Solution

P(Jack is in top 10%) implies that 90% of the area under normal curve is to left of his test score.

$\therefore$ Area = 0.9000

This area reflects a z-score of 1.29.

$$1.29 = \frac{75 - 65}{\sigma}$$

$$\sigma = \frac{10}{1.29} \cong 7.75$$

$$\sigma = 7.75$$

4. Pollsters estimate that the number of decided voters in favour of a particular bylaw is 64% and the number opposed is 36%.

a. If the sample size is 250, what is the expected mean and standard deviation of voters in favour of the bylaw?

b. Estimate, for this sample, the expected percentage of voters in favour of the bylaw in a symmetric 95% confidence interval.

Solution

a. $\mu = n \times p = 250 \times 0.64 = 160$

$\sigma = \sqrt{np(1-p)} = \sqrt{250(0.64)(0.36)} \doteq 7.59$

b. Because np = 160 ≥ 5 and n(1 – p) = 90 ≥ 5, the sample is large enough to use the normal approximation.

$$z_1 = -1.96 \qquad z_2 = 1.96$$

$$-1.96 = \frac{x - 160}{7.59} \qquad 1.96 = \frac{x - 160}{7.59}$$

$$-1.96(7.59) + 160 = x \qquad 7.59(1.96) + 160 = x$$

$$145.125 = x \qquad 174.875 = x$$

$$\frac{145.125}{250} \doteq 0.580499 \qquad \frac{174.875}{250} \doteq 0.699501$$

$\therefore$ Percentage of voters in favour of the bylaw expected is from 58.0% to 70.0%. Notice that the lower limit is rounded down and the upper limit is rounded up.

SE **5.** The results of a test were normally distributed with a mean of 57% and a standard deviation of 16%. The marks need to be adjusted so that the mean is 60% with a standard deviation of 12%. How can this be done?

Solution

Step 1 Add 3% to each test result, x. Now $\mu = 60$.

Step 2 Calculate z-scores for all these new test results by using $\mu = 60$ and $\sigma = 16$.

Step 3 Use these z-scores with $\mu = 60$ and $\sigma = 12$ to produce new test results (x values).

For example, if the original mark is 65%.

Step 1 mark becomes $65 + 3 = 68\%$.

Step 2 $z = \dfrac{68 - 60}{16} = 0.5$

Step 3 $0.5 = \dfrac{x - 60}{12}$

$x = 66\%$

The new test mark is 66%.

Note: Students will notice that not much change occurs to marks near to the mean and that marks farther from the mean change the most.

SE **6.** A written test for a driver's licence consists of 25 multiple-choice questions, each with 4 choices. To pass the test, a driver must answer at least 13 questions correctly.

a. What is the probability of passing the test by guessing?

b. What is the probability of getting exactly 13 questions correct by guessing?

Solution

a.
$$\left.\begin{aligned} np &= 25 \times 0.25 = 6.25 > 5 \\ n(1-p) &= 25 \times 0.75 = 18.75 > 5 \end{aligned}\right\} \therefore \text{ large sample}$$

$\mu = np = 6.25$

$\sigma = \sqrt{np(1-p)} = \sqrt{25(0.25)(0.75)}$

$\sigma \doteq 2.165$

$x = 13$

$$z = \frac{13 - 6.25}{2.165} = 3.12$$

$$\begin{aligned} P(x \geq 13) &= P(z \geq 3.12) \\ &= 1 - 0.9991 \\ &= 0.0009 \end{aligned}$$

Note: Because of the size of the z-score, this area calculation may need to be done with a different z-score table or on a graphing calculator.

b. $P(13) = {}_{25}C_{13}(0.25)^{13}(0.75)^{12} = 0.00245$

Note: $P(x = 13) = P(z = 3.12)$ cannot be calculated from the z-score table without the continuity correction. Since this is not an expected outcome, this question is dealt with using the binomial distrubution.

Pure Mathematics 30 Formula Sheet

For $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For two points (x_1, y_1) and (x_2, y_2),

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Graphing Calculator Window Format

x: $[x_{min}, x_{max}, x_{scl}]$

y: $[y_{min}, y_{max}, y_{scl}]$

Exponents, Logarithms, and Geometric Series

$$\log_a(M \times N) = \log_a M + \log_a N$$

$$\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N$$

$$\log_a M^n = n \log_a M$$

$$\log_b c = \frac{\log_a c}{\log_a b}$$

$$t_n = ar^{n-1}$$

$$S_n = \frac{a(r^n - 1)}{r - 1}, \quad r \neq 1$$

$$S_n = \frac{rt_n - a}{r - 1}, \quad r \neq 1$$

$$S = \frac{a}{1 - r}, \quad |r| < 1$$

Statistics

$$\mu = np \qquad \sigma = \sqrt{np(1 - p)}$$

$$z = \frac{x - \mu}{\sigma}$$

If $np \geq 5$ **and** $n(1 - p) \geq 5$, then the binomial distribution is a large sample.

Probability

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

$$P(A \text{ and } B) = P(A) \times P(B|A)$$

$$P(k) = {}_nC_k \; p^k(1 - p)^{n-k}$$

Permutations and Combinations

$${}_nP_r = \frac{n!}{(n - r)!}$$

$${}_nC_r = \frac{n!}{(n - r)!r!}$$

In the expansion of $(x + y)^n$, the general term is $t_{k+1} = {}_nC_k \; x^{n-k}y^k$.

Trigonometry

$$a = r\theta$$

$$\csc x = \frac{1}{\sin x}$$

$$\sec x = \frac{1}{\cos x}$$

$$\cot x = \frac{\cos x}{\sin x}$$

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x$$

$$\sin(A + B) = \sin A \cos B + \sin B \cos A$$

$$\sin(A - B) = \sin A \cos B - \sin B \cos A$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

$$\sin(2A) = 2 \sin A \cos A$$

$$\cos(2A) = \cos^2 A - \sin^2 A$$

Conics

General Form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

Standard Form

$$(x - h)^2 + (y - k)^2 = r^2$$

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = +1$$

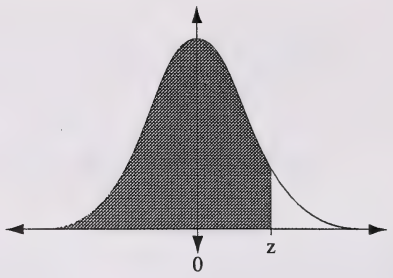
$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = \pm 1$$

$$y - k = a(x - h)^2$$

$$x - h = a(y - k)^2$$

Pure Mathematics z-Score Table

$$z = \frac{x - \mu}{\sigma}$$



Areas under the Standard Normal Curve

z	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01	0.00
–3.4	0.0002	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003
–3.3	0.0003	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0005	0.0005	0.0005
–3.2	0.0005	0.0005	0.0005	0.0006	0.0006	0.0006	0.0006	0.0006	0.0007	0.0007
–3.1	0.0007	0.0007	0.0008	0.0008	0.0008	0.0008	0.0009	0.0009	0.0009	0.0010
–3.0	0.0010	0.0010	0.0011	0.0011	0.0011	0.0012	0.0012	0.0013	0.0013	0.0013
–2.9	0.0014	0.0014	0.0015	0.0015	0.0016	0.0016	0.0017	0.0018	0.0018	0.0019
–2.8	0.0019	0.0020	0.0021	0.0021	0.0022	0.0023	0.0023	0.0024	0.0025	0.0026
–2.7	0.0026	0.0027	0.0028	0.0029	0.0030	0.0031	0.0032	0.0033	0.0034	0.0035
–2.6	0.0036	0.0037	0.0038	0.0039	0.0040	0.0041	0.0043	0.0044	0.0045	0.0047
–2.5	0.0048	0.0049	0.0051	0.0052	0.0054	0.0055	0.0057	0.0059	0.0060	0.0062
–2.4	0.0064	0.0066	0.0068	0.0069	0.0071	0.0073	0.0075	0.0078	0.0080	0.0082
–2.3	0.0084	0.0087	0.0089	0.0091	0.0094	0.0096	0.0099	0.0102	0.0104	0.0107
–2.2	0.0110	0.0113	0.0116	0.0119	0.0122	0.0125	0.0129	0.0132	0.0136	0.0139
–2.1	0.0143	0.0146	0.0150	0.0154	0.0158	0.0162	0.0166	0.0170	0.0174	0.0179
–2.0	0.0183	0.0188	0.0192	0.0197	0.0202	0.0207	0.0212	0.0217	0.0222	0.0228
–1.9	0.0233	0.0239	0.0244	0.0250	0.0256	0.0262	0.0268	0.0274	0.0281	0.0287
–1.8	0.0294	0.0301	0.0307	0.0314	0.0322	0.0329	0.0336	0.0344	0.0351	0.0359
–1.7	0.0367	0.0375	0.0384	0.0392	0.0401	0.0409	0.0418	0.0427	0.0436	0.0446
–1.6	0.0455	0.0465	0.0475	0.0485	0.0495	0.0505	0.0516	0.0526	0.0537	0.0548
–1.5	0.0559	0.0571	0.0582	0.0594	0.0606	0.0618	0.0630	0.0643	0.0655	0.0668
–1.4	0.0681	0.0694	0.0708	0.0721	0.0735	0.0749	0.0764	0.0778	0.0793	0.0808
–1.3	0.0823	0.0838	0.0853	0.0869	0.0885	0.0901	0.0918	0.0934	0.0951	0.0968
–1.2	0.0985	0.1003	0.1020	0.1038	0.1056	0.1075	0.1093	0.1112	0.1131	0.1151
–1.1	0.1170	0.1190	0.1210	0.1230	0.1251	0.1271	0.1292	0.1314	0.1335	0.1357
–1.0	0.1379	0.1401	0.1423	0.1446	0.1469	0.1492	0.1515	0.1539	0.1562	0.1587
–0.9	0.1611	0.1635	0.1660	0.1685	0.1711	0.1736	0.1762	0.1788	0.1814	0.1841
–0.8	0.1867	0.1894	0.1922	0.1949	0.1977	0.2005	0.2033	0.2061	0.2090	0.2119
–0.7	0.2148	0.2177	0.2206	0.2236	0.2266	0.2296	0.2327	0.2358	0.2389	0.2420
–0.6	0.2451	0.2483	0.2514	0.2546	0.2578	0.2611	0.2643	0.2676	0.2709	0.2743
–0.5	0.2776	0.2810	0.2843	0.2877	0.2912	0.2946	0.2981	0.3015	0.3050	0.3085
–0.4	0.3121	0.3156	0.3192	0.3228	0.3264	0.3300	0.3336	0.3372	0.3409	0.3446
–0.3	0.3483	0.3520	0.3557	0.3594	0.3632	0.3669	0.3707	0.3745	0.3783	0.3821
–0.2	0.3859	0.3897	0.3936	0.3974	0.4013	0.4052	0.4090	0.4129	0.4168	0.4207
–0.1	0.4247	0.4286	0.4325	0.4364	0.4404	0.4443	0.4483	0.4522	0.4562	0.4602
–0.0	0.4641	0.4681	0.4721	0.4761	0.4801	0.4840	0.4880	0.4920	0.4960	0.5000

Areas under the Standard Normal Curve

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Explanation of Cognitive Levels

Procedures

The assessment of students' knowledge of mathematical procedures should involve recognition, defense, execution and verification of appropriate procedures and the steps contained within them. The use of technology can allow for conceptual understanding prior to specific skill development. Students must appreciate that procedures are created or generated to meet specific needs in an efficient manner, and thus can be modified or extended to fit new situations. Assessment of students' procedural knowledge will not be limited to an evaluation of their proficiency in performing procedures, but will be extended to reflect the skills presented above.

In some instances to ensure fairness to all students regardless of their choice of calculators, certain types of procedural execution will not be tested on diploma examinations. This procedural execution is, however, an integral part of the *Program of Studies* and should be tested in the classroom.

Concepts

An understanding of mathematical concepts goes beyond a mere recall of definitions and recognition of common examples. Assessment of students' knowledge and understanding of mathematical concepts should provide evidence that they can compare, contrast, label, verbalize and define concepts, identify and generate examples and non-examples as well as properties of a given concept, and recognize the various meanings and interpretations of concepts. Students who have developed a conceptual understanding of mathematics can also use models, symbols and diagrams to represent concepts. Appropriate assessment will also provide evidence of the extent to which students have integrated their knowledge of various concepts.

Problem Solving

Appropriate assessment of problem solving skills is achieved by allowing students to adapt and extend the mathematics they know, and encouraging the use of strategies to solve unique and unfamiliar problems.

"Problem solving can be used effectively as a context in which students will learn new concepts and skills" *(Hiebert, 1998)*.

Assessment of problem solving involves measuring the extent to which students use these strategies and knowledge, and their ability to verify and interpret results. Students' ability to solve problems develops over time as a result of their experience with relevant situations that present opportunities to solve various types of problems.

Evidence of problem solving skills is often linked to clarity of communication. Students demonstrating strong problem-solving skills should be able to clearly explain the process they have chosen, using clear language and appropriate mathematical notation and conventions.

Calculator Policy and Calculator Clearing Instructions

Improvements in technology, and its increased availability in schools, have changed the focus of mathematics education. The time saved by using calculators and computers to perform complex calculations can be used to help students better understand mathematical concepts. Students can then understand the relationships among concepts and use these relationships to solve problems (*Applied and Pure Mathematics Program of Studies*, page 9).

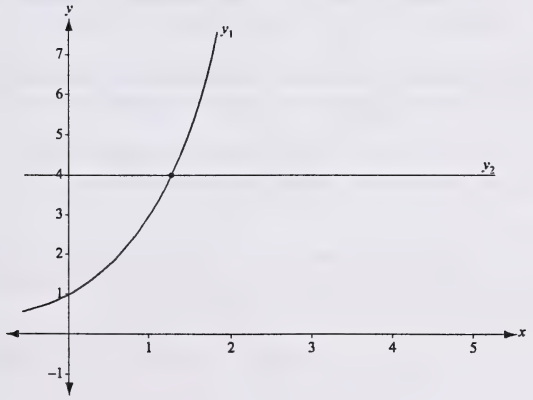
It is expected that students writing the Pure Mathematics 30 diploma examination will have an approved graphing calculator and be proficient in its use (see *Calculator Policy* for a list of approved calculators). If students are asked to provide a graph in their solution, they will be required to provide the dimensions of the graphing window used. The dimensions are to be communicated in the following manner.

x: $[x_{min}, x_{max}, x_{scl}]$ $\quad\quad$ y: $[y_{min}, y_{max}, y_{scl}]$

Some questions will require the use of a graphing calculator.

e.g. Using the window x:[0, 5, 1] y:[−1, 7, 1],
graph $y_1 = 3^x$ and $y_2 = 4$.
Determine, graphically, the x value at the intersection point.

Solution:



x is approximately 1.26, by using the Trace or Intersect function.

Alberta Learning Diploma Examinations Program Calculator Policy for Examinations Written Beginning in the School Year 2000–01

Rationale

The new Alberta Learning *Program of Studies* for mathematics began implementation with Grade 10 Mathematics in September 1998. It encourages teachers to integrate the use of technology into their lessons. This use of technology will assist teachers in emphasizing learning activities that develop high-level thinking skills and that allow students to solve complex, multifaceted problems. The *Mathematics Applied and Pure Program of Studies* states:

> *Improvements in technology, and its increased availability in schools, have changed the focus of mathematics education. The time saved by using calculators or computers to perform complex calculations can be used to help students better understand mathematical concepts. Students can then understand the relationships among concepts and use these relationships to solve problems.*
>
> *Calculators and computers can be used as tools to*
> - *explore and demonstrate mathematical relationships and patterns*
> - *organize and display data*
> - *assist with solving problems*
> - *decrease the time spent on tedious computations*
> - *simulate situations*

Professional mathematics organizations such as the National Council of Teachers of Mathematics (NCTM) and the Mathematics Council, Alberta Teachers' Association (MCATA), have strongly endorsed the use of graphing calculators in mathematical instruction. They also emphasize that if calculator and computer technologies are now accepted as part of the environment in which students learn and do mathematics, such tools should also be available to students in most assessment situations.

As a result, the calculator policy for the Pure Mathematics 30 and Applied Mathematics 30 diploma examinations, includes the use of graphing calculators. Although many questions will not require the use of a graphing calculator, there will be some questions for which the use of a graphing calculator will be required.

All science diploma examinations and examinations in Mathematics 30 (old program) and Mathematics 33 will require the use of a scientific calculator or of a graphing calculator approved by Alberta Learning.

Note: This policy applies to Alberta Learning diploma examinations and field tests only. School calculator policies may differ depending on grade level or topic studied.

Definition

The calculator must be a hand-held device designed primarily for mathematical computations, including logarithmic and trigonometric functions, as well as for graphing functions. Included in this definition are those scientific calculators having graphing and programmable features.

Policy

To ensure compatibility with the provincial *Programs of Study* and equity and fairness to all students, Alberta Learning expects students to use calculators, as defined, when writing diploma examinations in mathematics and the sciences. **Pure Mathematics 30 and Applied Mathematics 30 require the use of an approved graphing calculator for diploma examinations. All science diploma examinations and examinations in Mathematics 30 (Old Program) and Mathematics 33 require the use of a scientific calculator or an approved graphing calculator.**

Procedures

1. At the beginning of **any** mathematics or science diploma examination course, teachers must advise students of the types of calculators approved by Alberta Learning for use when writing diploma examinations in these courses.
2. Students must clear **all** programmable calculators, both graphing and scientific, that are brought into diploma examinations of all information that is stored in the programmable or parametric memory.
3. Students must not bring external devices (peripherals) to support calculators into any examination. Such devices include manuals, printed or electronic cards, printers, memory expansion chips or cards, external keyboards, CD-ROMs, libraries, or any annotations that outline operational procedures.
4. In preparation for calculator failure, students may bring extra batteries and/or approved calculators into the examination room.
5. During examinations, supervising teachers must ensure that
   - calculators operate in silent mode
   - students do not share calculators or information contained within them
   - calculator cases are not available to students
   - programmable calculator memories, including parametric memories, have been cleared
   - only graphing calculators on the current list approved by Alberta Learning are used

Calculator Criteria

The following criteria will be used to select acceptable calculators.

Minimum calculator properties required

1. Function graphing capabilities with display
   —includes displaying more than one function on the screen at a time, tracing a function
2. Standard scientific calculator operations
   —e.g., sine, cosine, tangent, inverse functions, logarithms, power (x^n)
3. Statistical functions in 1 and 2 variables
   —mean, median, mode, standard deviation, bivariate data, regression models
4. List capabilities
5. Matrix capabilities
   —scalar multiplication, addition, and subtraction

Unacceptable calculator properties during examinations

1. Built-in notes (definitions or explanations in alpha notation), e.g., libraries
2. Upgrades that include built-in notes
3. Remote communication ability

The following list of approved calculators is provided to assist students and teachers in the selection of graphing calculators that conform to the requirements stated in the definition and to the stated criteria. The list will be updated annually.

Note: All the calculators listed below meet the "required properties". They do not have any "unacceptable properties" and so can be used on the mathematics and science diploma examinations. However, students and teachers should recognize that the different models of calculators listed have a range of capabilities, and the choice of which model to use or purchase will require personal or teacher analysis of the machines' capabilities and one's individual or school circumstances.

The List of Approved Graphing Calculators

Brands	*Casio*	*Sharp*	*Hewlett-Packard*	*Texas Instruments*
Models	Algebra FX 2.0	El-9600C	HP39G*	TI-82* TI-83 TI-83 Plus TI-86 TI-89 TI-92 Plus

*** The TI-82 calculator will remain on the approved list for the 2001–2002 school year and will then be deleted from the approved list. The HP39G calculator will remain on the approved list for the 2001–2002 and 2002–2003 school years and then will be deleted from the approved list. These calculators are not recommended for students entering high school mathematics who wish to use a calculator throughout their high school program.**

The following calculators meet the graphing calculator criteria and are approved, but are no longer commercially manufactured.

Brands	*Casio*	*Sharp*	*Texas Instruments*
Models	FX-9700 series CFX-9800G CFX-9850G CFX-9850GA Plus	El-9600 El-9200 El-9300C	TI-92

Note: Instructions for clearing calculator memories are posted in the Alberta Learning web site: http://www.learning.gov.ab.ca/K_12/testing/diploma/bulletins/essential/clearing_calc.asp

For Further Information

If you have any questions or comments about this policy, please contact the Assistant Director of the Mathematics/Science Diploma Examination Unit, Learner Assessment Branch, at (780)427-0010, fax (780)422-4454. To call toll-free from outside of Edmonton, dial 310-0000.

Keystrokes Required For Clearing **Approved** Calculators

The following information is provided to help familiarize students, teachers, and presiding examiners with the procedures involved in clearing all calculator memories.

Procedures to be followed prior to the writing of a diploma examination.

1. At the beginning of **any** mathematics or science diploma examination course, teachers must advise students of the types of calculators approved by Alberta Learning for use when writing diploma examinations.
2. Students must clear **all** programmable calculators, both graphing and scientific, that are brought into diploma examinations. All information that is stored in the programmable or parametric memory must be cleared.
3. Presiding examiners are responsible for ensuring that
    a. all calculators operate in silent mode
    b. students do not share calculators or information contained within them
    c. calculator cases are stored on the floor throughout the examination
    d. all examination rules are followed

Note 1
If you find that there are problems with any of the clearing techniques, please contact the Assistant Director of the Mathematics/Science Diploma Examination Unit, Learner Assessment Branch, at (780)427-0010, fax (780)422-4454. To call toll-free from outside of Edmonton, dial 310-0000.

Note 2
Resetting calculators may result in altering the calculator mode settings. Please remember to check the mode settings before proceeding with the diploma examination.

Note 3
Programs downloaded from the web are not allowed on the calculators used during diploma examinations and will be erased by these procedures.

Note 4
The memory values given on the next pages refer to memory expected to be available as a factory setting. The values available in student calculators should match these values when the calculator has been reset. If the values in the student calculators do not match these values then the calculators should be reset a second time. If this fails to change the values, then the calculator should not be used on the examination.

Texas Instruments	Memory remaining
TI 82 **2nd** **+** (MEM) **3** (Reset) **2** (Reset) **Note: If, on clearing, the screen is blank, the contrast needs to be reset. To do this, use 2nd ⇑ both repeatedly.**	**2nd** **+** **1** **MEM FREE 28734**
TI 83 **2nd** **+** (MEM) **5** (Reset) **1** (All memory) **2** (Reset) **Note: If, on clearing, the screen is blank, the contrast needs to be reset. To do this, use 2nd ⇑ both repeatedly.**	**2nd** **+** **1** **RAM 27118**
TI 83 Plus **2nd** **+** (MEM) **7** (Reset) **➤➤** (All) **Enter** **2** (Reset) **Note: If, on clearing, the screen is blank, the contrast needs to be reset. To do this, use 2nd ⇑ both repeatedly.**	**2nd** **+** **2** **RAM 24317** **ARC 163840**
TI 86 **2nd** **3** (MEM menu) **F3** (Reset) **F1** (All) **F4** (Yes) **Note: If, on clearing, the screen is blank, the contrast needs to be reset. To do this, use 2nd ⇑ both repeatedly.**	**2nd** **3** **F1** **MEM FREE 98226**
TI 89 **2nd** **6** (MEM) **F1** (All) **1** (Reset) Enter	**2nd** **6** **RAM 199154** **ARC 393204**
TI 92 **2nd** **6** (MEM) **F1** (Reset) **1** (All) **Enter** **Note: If, on clearing, the screen is blank, the contrast needs to be reset. To do this, use ◊ (green) and + or – repeatedly.**	**2nd** **6** **System 61064** **Memory Free 70008**

Casio

Memory remaining

Casio FX9700 series Go to **Menu** Cursor to **Reset** **EXE** (All Memory) **F1** (Yes – Reset All)	**Menu** **24 000 Bytes available** **EXE** **Clear** **Shift** **MDISP** (CAPA)
Casio CFX9800G Go to **Menu** **Options** **EXE** (Memory) **Reset** **EXE** (Reset) **F1** (Yes – Reset All)	**Menu** **24 000 Bytes available** **EXE** **Clear** **Shift** **MDISP** (CAPA)
Casio CFX-9850G, CFX-9850GA, CFX-9850GA Plus Go to **Menu** Cursor to **Mem** **EXE** (Memory) ⇓ (Reset) **EXE** **F1** (Yes – Reset All) Note: For the GA plus, use Alpha F.	**Menu** **30677** **Alpha E** **EXE** **Check usage**
Casio Algebra FX 2.0 Go to **Menu** Cursor to **SYSTEM** **EXE** System manager **F5** Reset **F2** Reset (Clear) Main Memories? **EXE** Yes – Main Memories Cleared **ESC** Reset **F3** Reset (Clear) Storage Memories? **EXE** Test – Storage Memories Cleared **Menu** **Note: A tutorial system exists on this calculator and may be locked out for 180 minutes by following steps illustrated in the manual. Since the information in the tutorial has no effect upon the Pure 30/Applied 30 Mathematics topics, application of this process is not necessary. This may have implications for Pure 10-20/Applied 10-20topics.**	Go to **Menu** Cursor to **Memory** **EXE** **Memory Manager** **F1** **Current Area No Program (146912 Bytes Free.)** **F6** **Storage Area No Program (786240 Bytes Free.)** **Menu**

Sharp

	Memory remaining
Sharp EL 9600 and 9600 C **2nd** **XθTN** (Option) **Log** (Reset) **2** (All Memory) **CL** (Clear all data) **Note: There is also a reset switch on the back. (Use round tip of pen, press, then CL)**	**2nd** ⇓ **XθTN** **18562**

Hewlett-Packard

	Memory Remaining
Hewlett-Packard HP39G Hold down simultaneously: **ON** Menu Key 1 Menu Key 6 Release **OK** (Menu Key 6)	**SHIFT** , **Memory Manager 234 K** **Note:** All items in this window should read 0 KB.

Directing Words

Discuss
The word "discuss" will not be used as a directing word on math and science diploma examinations because it is not used consistently to mean a single activity

The following words are specific in meaning.

Algebraically
Using mathematical procedures

Compare
Show the character or relative values of two things by pointing out their *similarities* and *differences*

Conclude
State a logical end based on reasoning and/or evidence

Contrast/Distinguish
Point out the *differences* between two things that have similar or comparable natures

Define
Provide the essential qualities or meaning of a word or concept; make distinct and clear by marking out the limits

Describe
Give a written account or represent the characteristics of something by a figure, model, or picture

Design/Plan
Construct a plan, i.e, a detailed sequence of actions, for a specific purpose

Determine
Find a solution, to a specified degree of accuracy, to a problem by showing appropriate formulas, procedures, and calculations

Evaluate
Give the significance or worth of something by identifying the good and bad points or the advantages and disadvantages

Explain
Make clear what is not immediately obvious or entirely known; give the cause of or reason for; make known in detail

Graphically
Using a drawing, that is produced electronically or by hand, that shows a relation between certain sets of numbers **NEW**

How
Show in what manner or way, with what meaning

Hypothesize
Form a tentative proposition intended as a possible explanation for an observed phenomenon; i.e., a possible cause for a specific effect. The proposition should be testable logically and/or empirically

Identify
Recognize and select as having the characteristics of something

Illustrate
Make clear by giving an example. The form of the example must be specified in the question; i.e., word description, sketch, or diagram

Infer
Form a generalization from sample data; arrive at a conclusion by reasoning from evidence

Interpret
Tell the meaning of something; present information in a new form that adds meaning to the original data

Justify/Show How
Show reasons for or give facts that support a position

Model
Find a model (in mathematics, a model of a situation is a pattern that is supposed to represent or set a standard for a real situation) that does a good job of representing a situation.

Outline
Give, in an organized fashion, the essential parts of something. The form of the outline must be specified in the question; i.e., lists, flow charts, concept maps

Predict
Tell in advance on the basis of empirical evidence and/or logic

Prove
Establish the truth of validity of a statement for the general case by giving factual evidence or logical argument

Relate
Show logical or causal connection between things

Sketch **NEW**
Find points that lie on a curve, and draw through these points a curve which is assumed to resemble the required curve

Solve
Give a solution for a problem; i.e., explanation in words and/or numbers

Summarize
Give a brief account of the main points

Trace
Give a step-by-step description of the development

Verify
Establish, by substitution for a particular case or by geometric comparison, the truth of a statement

Why
Show the cause, reason, or purpose

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